

# Head-on infall of two compact objects: Third post-Newtonian Energy Flux

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Head-on infall of two compact objects with arbitrary mass ratio is investigated using the multipolar post-Minkowskian approximation method. At the third post-Newtonian order the energy flux, in addition to the instantaneous contributions, also includes hereditary contributions consisting of the gravitational-wave tails, tails-of-tails and the tail-squared terms. The results are given both for infall from infinity and also for infall from a finite distance. These analytical expressions should be useful for the comparison with the high accuracy numerical relativity results within the limit in which post-Newtonian approximations are valid.

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## I. INTRODUCTION

The spiraling coalescence of two compact objects (black holes or neutron stars) moving about one another in an orbit, forms a prominent class of sources of gravitational radiation [1]. Such sources of gravitational waves (GW), especially in their late stages of evolution are prime targets for gravitational wave detectors such as LIGO [2] and Virgo [3]. The evolution of the binary systems composed of two compact objects involves three stages of evolution; the early inspiral, late inspiral and merger and the final ringdown. Detection of gravitational radiation from such systems by the gravitational wave detectors depends strongly on the theoretical inputs, which will involve computation of the waveform of the signal for all the three phases to very high post-Newtonian (PN) order to detect and infer the characteristics of the sources of GWs, using matched filtering techniques [4]. Even though head-on collision of two black holes has only a small astrophysical possibility, it provides the simplest possible situation to study the two-body problem of general relativity and has been studied since it provides an excellent theoretical platform for comparing the validity of various analytical and numerical approaches towards solving Einstein's equations in dynamical situations.

One of the earliest attempts to solve the problem of head-on collision using a complete general relativistic approach was due to Davis et al [5]. They discussed the emission of gravitational radiation due to the radial infall of a test particle in Schwarzschild spacetime from infinity, using Zerilli's equation for black-hole perturbations [6]. Because of the axial symmetry of the system, the problem simplifies considerably and yet retains the features of astrophysical interest such as emission of gravitational radiation at infinity. In addition to this, head-on collision can be considered as an approximation to the last stage of the inspiralling coalescence-when two objects merge together to form a single object. The first attempt to solve the head-on collision of two equal mass black holes numerically was due to Smarr and Eppley [7–9]. This program has undergone substantial improvement in accuracy and reliability with advances in the understanding of numerical issues in the treatment of Einstein's equations and availability of better computing [10]. The head-on collision of two black holes with arbitrary mass ratio has been investigated numerically in [11, 12] and semianalytically [13]. In a recent work [14] head-on collision of two equal mass, nonrotating black holes with ultrarelativistic speeds have been studied using numerical methods. The main result of this analysis is that in such a process (where the initial energy of the system is dominated by kinetic energy of black holes) the total amount of energy converted to gravitational waves is about 14% of the initial mass-energy for the system and corresponds to large luminosities of the order of  $10^{-2} c^5/G$ . Another study related to the collision of two equal mass, nonrotating black holes moving at ultrarelativistic speeds and with generic impact parameter [15] suggests that such collisions can produce black holes rotating close to the Kerr limit and the energy radiated in such a process would be roughly 35% of the center-of-mass (CM) energy.

Another approach which may be used to study the head-on collision of two compact objects is the PN approximation approach. Though PN methods are valid for arbitrary mass ratios, they eventually break down under situations like strong gravitational fields and high speeds. Simone, Poisson and Will (SPW) [16] investigated the problem of head-on infall and compared the PN approach with black-hole perturbation (BHP) theory. They provided 2PN accurate

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expression for the far-zone GW energy flux and showed, in particular, that the energy radiated during the infall is well-estimated by the quadrupole approximation combined with the *exact* test-body equations of motion (EOM) in Schwarzschild background. Also in a recent study [17], a hybrid method using both PN approximations and BHP theories has been used to study the head-on collision of two black holes and found that PN and BHP theories can explain the main features of gravitational radiation for head-on mergers.

In this paper we investigate the problem of head-on infall using the multipolar post-Minkowskian (MPM) approach [18–23] and provide the complete 3PN accurate expression for the GW energy flux emitted during the radial infall of two compact objects towards each other. In addition to the simpler instantaneous part of the energy flux we also compute the more complex hereditary contributions up to 3PN order which involves the contributions due to tails, tails-of-tails and tail-squared terms. We discuss the head-on problem both for infall starting from rest at an initial finite separation (denoted case I) and similarly for infall starting from rest at infinite separation (denoted by case II). Instantaneous contributions at 2.5PN order and at 3PN order, computation of tails at 2.5PN, tail-of-tail and tail-squared terms at 3PN order are the new results of this paper. Our computations suggest that the total energy radiated in the process of head-on infall of two compact objects with equal masses is roughly about 0.0074% of the Arnowitt, Deser, and Misner (ADM) mass of the binary and the peak luminosities are typically less than of the order  $5 \times 10^{-6} c^5/G$ . Comparing our PN estimates with the numerical relativity results [10] we can see that the PN estimates are smaller than the numerical results typically by a factor of 27 consistent with the expectation that a larger fraction of energy radiated indeed comes from the merger phase of the infall rather than from the early inspiral.

This paper is organized in the following way. In Sec. II we begin by providing the structure of the far-zone GW energy flux at 3PN order, relations connecting radiative multipole moments to source multipole moments and the decomposition of the expression for energy flux into instantaneous and hereditary contributions. Section III lists the 3PN EOM as well as the 3PN accurate expression for the center-of-mass energy in standard harmonic coordinates for the head-on case. In Sec. IV we give the expressions for the desired multipole moments at the PN order required for the computation of 3PN energy flux for head-on infall case. In Sec. V we first exhibit the instantaneous part of energy flux up to 3PN order in standard harmonic coordinates followed by the corresponding expressions in two alternative coordinates for possible comparison with numerical relativity results: modified harmonic (MH) and ADM. Section VI describes the computation of the hereditary part of the energy flux. Finally, in Sec. VII, we bring together the complete 3PN accurate expression for energy flux in ADM coordinates and the energy radiated during infall to some fixed radial coordinate. Section VIII contains a graphical display of the salient features and our conclusions. These results should be useful to compare and match to simulations using numerical methods in regimes where both treatments are expected to be the valid. The paper ends with a short appendix relating the expression for conserved energy in standard harmonic (SH) coordinates to that in ADM coordinates.

## II. THE FAR-ZONE GW ENERGY FLUX

We start the discussion by writing the 3PN expression for far-zone GW energy flux in terms of the symmetric trace-free radiative multipole moments [24, 25]. The PN structure for GW energy flux reads as,

$$\begin{aligned} \mathcal{F}(U) = \frac{G}{c^5} & \left\{ \frac{1}{5} U_{ij}^{(1)} U_{ij}^{(1)} \right. \\ & + \frac{1}{c^2} \left[ \frac{1}{189} U_{ijk}^{(1)} U_{ijk}^{(1)} + \frac{16}{45} V_{ij}^{(1)} V_{ij}^{(1)} \right] + \frac{1}{c^4} \left[ \frac{1}{9072} U_{ijkm}^{(1)} U_{ijkm}^{(1)} + \frac{1}{84} V_{ijk}^{(1)} V_{ijk}^{(1)} \right] \\ & \left. + \frac{1}{c^6} \left[ \frac{1}{594000} U_{ijkmn}^{(1)} U_{ijkmn}^{(1)} + \frac{4}{14175} V_{ijkm}^{(1)} V_{ijkm}^{(1)} \right] + \mathcal{O}(8) \right\}. \end{aligned} \quad (2.1)$$

In the above expression  $U_L$  and  $V_L$  (where  $L = i_1 i_2 \dots i_l$  represents a multi-index composed of  $l$  spatial indices) are the mass-type and current-type radiative multipole moments respectively and  $U_L^{(n)}$  and  $V_L^{(n)}$  denote their  $n^{\text{th}}$  time derivatives. The moments appearing in the formula are functions of retarded time  $U \equiv T - R/c$  in radiative coordinates.

Equation (2.1) is the general formula for the computation of 3PN accurate energy flux for any general isolated source. In a recent paper [25] the complete third post-Newtonian energy flux has been computed for inspiralling compact binaries moving in quasi-elliptical orbits. In the present work we specialize to the case of head-on infall and compute the 3PN accurate far-zone GW energy flux emitted due to head-on infall of two compact objects with arbitrary mass ratio using the MPM approximation method. The radiative current-type moments ( $V_L$ ) are related to the source current moments  $J_L$  whose expansion at each PN order contains the orbital angular momentum  $\mathcal{J}$  which

vanishes in the head-on case. Thus the current-type moments  $V_L$  will not contribute to GW energy flux and for the head-on case, Eq. (2.1) reduces to the following form,

$$\mathcal{F}(U) = \frac{G}{c^5} \left\{ \frac{1}{5} U_{ij}^{(1)} U_{ij}^{(1)} + \frac{1}{c^2} \left[ \frac{1}{189} U_{ijk}^{(1)} U_{ijk}^{(1)} \right] + \frac{1}{c^4} \left[ \frac{1}{9072} U_{ijkm}^{(1)} U_{ijkm}^{(1)} \right] + \frac{1}{c^6} \left[ \frac{1}{594000} U_{ijkmn}^{(1)} U_{ijkmn}^{(1)} \right] + \mathcal{O}(8) \right\}. \quad (2.2)$$

In the MPM formalism, the radiative moments  $U_L$  and  $V_L$  are related to canonical moments  $M_L$  and  $S_L$  respectively and these canonical moments are in turn expressed in terms of source moments  $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$ . Since in the present work we only deal with head-on situation we would exclude terms involving current-type multipole moments from all our expressions for the reason stated above. It should be evident from the Eq. (2.2) that for the computation of 3PN accurate energy flux  $U_{ij}$  is needed at 3PN order,  $U_{ijk}$  is needed at 2PN order,  $U_{ijkl}$  with 1PN accuracy and  $U_{ijklm}$  to leading Newtonian accuracy. General expressions connecting  $U_L$  to source moments have been listed in [25] and we shall simply recall those expressions. For the 3PN accurate mass quadrupole we have

$$\begin{aligned} U_{ij}(U) = & I_{ij}^{(2)}(U) + \frac{2GM}{c^3} \int_0^{+\infty} d\tau \left[ \log \left( \frac{c\tau}{2r_0} \right) + \frac{11}{12} \right] I_{ij}^{(4)}(U - \tau) \\ & + \frac{G}{c^5} \left\{ -\frac{2}{7} \int_0^{+\infty} d\tau I_{a(i}^{(3)}(U - \tau) I_{j)a}^{(3)}(U - \tau) \right. \\ & + \frac{1}{7} I_{a(i}^{(5)} I_{j)a} - \frac{5}{7} I_{a(i}^{(4)} I_{j)a}^{(1)} - \frac{2}{7} I_{a(i}^{(3)} I_{j)a}^{(2)} \\ & \left. + 4 \left[ W^{(2)} I_{ij} - W^{(1)} I_{ij}^{(1)} \right]^{(2)} \right\} \\ & + 2 \left( \frac{GM}{c^3} \right)^2 \int_0^{+\infty} d\tau I_{ij}^{(5)}(U - \tau) \left[ \log^2 \left( \frac{c\tau}{2r_0} \right) + \frac{57}{70} \log \left( \frac{c\tau}{2r_0} \right) + \frac{124627}{44100} \right] + \mathcal{O}(7), \end{aligned} \quad (2.3)$$

where the bracket  $\langle \rangle$  surrounding indices denotes the symmetric trace-free projection. The  $I_L$ 's are the mass-type source moments (and  $I_L^{(n)}$  denote their  $n^{\text{th}}$  time derivatives), and  $W$  is the monopole corresponding to the gauge moment  $W_L$  which for our purpose needs to be known Newtonian accuracy. The quantity  $M$  appearing in the above expression is the ADM mass of the source. It should be evident from Eq. (2.3) that radiative moments have two distinct contributions. The first referred to as the *instantaneous* contribution requires the knowledge of source multipole moments only at a given retarded time,  $U = T - R/c$ ; where  $R$  is the distance of the source in radiative coordinates. The second one, referred to as the *hereditary* contribution, which is given by integrals over retarded time from 0 to  $\infty$ , depends on the dynamics of the system in its entire past history and requires the knowledge of source moments at all times before  $U$ . A closer look at the hereditary terms reveals two types of contributions, some with and some without the log factors. The integrals (with log factors) appearing at 1.5PN and 3PN order are called tail and tail-of-tail integrals respectively. The integral (without log factor) appearing at 2.5PN order is called the nonlinear memory integral. It is a time antiderivative and hence leads to an instantaneous term in the energy flux.

The mass-type octupole moment  $U_{ijk}$  which is needed at 2PN is related to the associated source moment as

$$U_{ijk}(U) = I_{ijk}^{(3)}(U) + \frac{2GM}{c^3} \int_0^{+\infty} d\tau \left[ \log \left( \frac{c\tau}{2r_0} \right) + \frac{97}{60} \right] I_{ijk}^{(5)}(U - \tau) + \mathcal{O}(5). \quad (2.4)$$

For other radiative moments,  $U_{ijkl}$  and  $U_{ijklm}$ , only the leading order accuracy in the relation between radiative and source moments is needed, so that

$$U_L(U) = I_L^{(l)}(U) + \mathcal{O}(3). \quad (2.5)$$

The constant  $r_0$  which provides a scale for the logarithmic term in the above expressions is an arbitrary constant. It enters the relation connecting retarded time  $U = T - R/c$  in radiative coordinates to retarded time  $u = t - r/c$  in harmonic coordinates (where  $r$  is the distance of the source in harmonic coordinates). The relation between retarded time in radiative coordinates, and the one in harmonic coordinates reads as

$$U = t - \frac{r}{c} - \frac{2GM}{c^3} \log \left( \frac{r}{r_0} \right) + \mathcal{O}(5). \quad (2.6)$$

Later in this paper we shall show that the presence of this constant  $r_0$  will not influence any physical result like far-zone GW energy flux.

We can now use the expressions for the radiative moments given by Eqs. (2.3)-(2.5) in Eq. (2.2) to obtain the 3PN energy flux formula in terms of source moments. As discussed above, the presence of two distinct contributions (instantaneous and hereditary) leads to a natural decomposition of the 3PN energy flux into two pieces and the complete flux can be written as a sum of the two distinct types of contributions as

$$\mathcal{F} = \mathcal{F}_{\text{inst}} + \mathcal{F}_{\text{hered}}, \quad (2.7)$$

where the instantaneous contribution<sup>1</sup> to the energy flux is given by

$$\begin{aligned} \mathcal{F}_{\text{inst}}(U) = & \frac{G}{c^5} \left\{ \frac{1}{5} I_{ij}^{(3)} I_{ij}^{(3)} + \frac{1}{c^2} \left[ \frac{1}{189} I_{ijk}^{(4)} I_{ijk}^{(4)} \right] + \frac{1}{c^4} \left[ \frac{1}{9072} I_{ijkm}^{(5)} I_{ijkm}^{(5)} \right] \right. \\ & + \frac{G}{c^5} \left[ \frac{8}{5} I_{ij}^{(3)} \left( I_{ij} W^{(5)} + 2 I_{ij}^{(1)} W^{(4)} - 2 I_{ij}^{(3)} W^{(2)} - I_{ij}^{(4)} W^{(1)} \right) \right. \\ & \quad \left. + \frac{2}{5} I_{ij}^{(3)} \left( -\frac{4}{7} I_{ai}^{(5)} I_{aj}^{(1)} - I_{ai}^{(4)} I_{aj}^{(2)} - \frac{4}{7} I_{ai}^{(3)} I_{aj}^{(3)} + \frac{1}{7} I_{ai}^{(6)} I_{aj} \right) \right] \\ & \left. + \frac{1}{c^6} \left[ \frac{1}{594000} I_{ijkmn}^{(6)} I_{ijkmn}^{(6)} \right] + \mathcal{O}(7) \right\}. \end{aligned} \quad (2.8)$$

The hereditary contribution comprises of three parts,

$$\mathcal{F}_{\text{hered}} = \mathcal{F}_{\text{tail}} + \mathcal{F}_{\text{tail}(\text{tail})} + \mathcal{F}_{(\text{tail})^2}, \quad (2.9)$$

where the quadratic-order (proportional to  $G^2$ ) tails are given by

$$\begin{aligned} \mathcal{F}_{\text{tail}}(U) = & \frac{4 G^2 M}{5 c^8} I_{ij}^{(3)}(U) \int_0^{+\infty} d\tau I_{ij}^{(5)}(U - \tau) \left[ \log \left( \frac{c\tau}{2r_0} \right) + \frac{11}{12} \right] \\ & + \frac{4 G^2 M}{189 c^{10}} I_{ijk}^{(4)}(U) \int_0^{+\infty} d\tau I_{ijk}^{(6)}(U - \tau) \left[ \log \left( \frac{c\tau}{2r_0} \right) + \frac{97}{60} \right], \end{aligned} \quad (2.10)$$

and the cubic-order tails (proportional to  $G^3$ ) by

$$\mathcal{F}_{\text{tail}(\text{tail})}(U) = \frac{4 G^3 M^2}{5 c^{11}} I_{ij}^{(3)}(U) \int_0^{+\infty} d\tau I_{ij}^{(6)}(U - \tau) \left[ \log^2 \left( \frac{c\tau}{2r_0} \right) + \frac{57}{70} \log \left( \frac{c\tau}{2r_0} \right) + \frac{124627}{44100} \right], \quad (2.11a)$$

$$\mathcal{F}_{(\text{tail})^2}(U) = \frac{4 G^3 M^2}{5 c^{11}} \left( \int_0^{+\infty} d\tau I_{ij}^{(5)}(U - \tau) \left[ \log \left( \frac{c\tau}{2r_0} \right) + \frac{11}{12} \right] \right)^2. \quad (2.11b)$$

Here one should note that the general formulae for energy flux include some contributions from current-type moments as well (see [25, 26]) but these vanish for the head-on case. Further, it should be noted that Eqs. (2.3)-(2.4) and thus Eqs. (2.10)-(2.11) show an intermediate dependence on the arbitrary length scale  $r_0$  which should eventually cancel from all physical quantities. Such a cancellation of the scale  $r_0$  from all physical quantities occurs naturally in the MPM formalism and has been explicitly shown for sources such as binary systems moving in circular [20] and elliptical orbits [26, 27]. This is facilitated because an explicit computation of the hereditary integrals is possible since the integral over the complete past in the adiabatic approximation reduces to an integral over the current (noninspiralling) orbit which can then be computed making explicit use of the periodicity features in the motion. For a head-on situation, on the other hand, the absence of periodicity prevents the straightforward extension of the above method. A more first-principle treatment is called for based on the observation that since most of the  $r_0$ -dependence comes from our definition [Eq. (2.6)] of a radiative coordinate system, it can be tracked and isolated by inserting  $U$  as given by Eq. (2.6) back in Eq. (2.3)-(2.4). Upon doing so we get expressions for radiative moments in harmonic coordinates  $(t, r)$ , which read

$$U_{ij}(u) = I_{ij}^{(2)}(u) + \frac{2 G M}{c^3} \int_0^{+\infty} d\tau \left[ \log \left( \frac{c\tau}{2r} \right) + \frac{11}{12} \right] I_{ij}^{(4)}(u - \tau)$$

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<sup>1</sup> There is a typographical error in Eq. (2.7) of [25] which has been corrected while writing Eq. (2.8) of the present work. At 2.5PN order the coefficient of  $I_{ij}^{(3)} I_{ij}^{(3)}$  should be  $-\frac{4}{7}$  and not  $-\frac{2}{7}$ . However, the results in [25] are computed using the correct coefficient.

$$\begin{aligned}
& + \frac{G}{c^5} \left\{ -\frac{2}{7} \int_0^{+\infty} d\tau I_{a\langle i}^{(3)}(u-\tau) I_{j\rangle a}^{(3)}(u-\tau) \right. \\
& \quad + \frac{1}{7} I_{a\langle i}^{(5)} I_{j\rangle a} - \frac{5}{7} I_{a\langle i}^{(4)} I_{j\rangle a}^{(1)} - \frac{2}{7} I_{a\langle i}^{(3)} I_{j\rangle a}^{(2)} \\
& \quad \left. + 4 \left[ W^{(2)} I_{ij} - W^{(1)} I_{ij}^{(1)} \right]^{(2)} \right\} \\
& + 2 \left( \frac{GM}{c^3} \right)^2 \int_0^{+\infty} d\tau I_{ij}^{(5)}(u-\tau) \left[ \log^2 \left( \frac{c\tau}{2r} \right) + \frac{57}{70} \log \left( \frac{c\tau}{2r} \right) + \frac{124627}{44100} \right] \\
& - \frac{214}{105} \log \left( \frac{r}{r_0} \right) \left( \frac{GM}{c^3} \right)^2 I_{ij}^{(4)}(u) + \mathcal{O}(7), \tag{2.12}
\end{aligned}$$

$$U_{ijk}(u) = I_{ijk}^{(3)}(u) + \frac{2GM}{c^3} \int_0^{+\infty} d\tau \left[ \log \left( \frac{c\tau}{2r} \right) + \frac{97}{60} \right] I_{ijk}^{(5)}(u-\tau) + \mathcal{O}(5). \tag{2.13}$$

In the 1.5PN term, the  $r_0$  dependence is more trivial and disappears with the change from radiative to harmonic coordinates. At 3PN order, however, there still remains a nontrivial  $r_0$  dependent term. However, the quadrupole mass moment  $I_{ij}$  also depends on the constant  $r_0$  at 3PN order [see Eq. (4.2)] and when one takes those dependences into account we will check that the 3PN radiative moment  $U_{ij}$  is indeed independent of  $r_0$ . Using the expressions for radiative moments given by Eqs. (2.12) and (2.13) we can rewrite explicitly the different hereditary contributions given by Eqs. (2.9), (2.10), and (2.11). One finds the quadratic-order (proportional to  $G^2$ ) tails are given by

$$\begin{aligned}
\mathcal{F}_{\text{tail}}(u) &= \frac{4G^2 M}{5c^8} I_{ij}^{(3)}(u) \int_0^{+\infty} d\tau I_{ij}^{(5)}(u-\tau) \left[ \log \left( \frac{c\tau}{2r} \right) + \frac{11}{12} \right] \\
&+ \frac{4G^2 M}{189c^{10}} I_{ijk}^{(4)}(u) \int_0^{+\infty} d\tau I_{ijk}^{(6)}(u-\tau) \left[ \log \left( \frac{c\tau}{2r} \right) + \frac{97}{60} \right], \tag{2.14}
\end{aligned}$$

and the cubic-order tails (proportional to  $G^3$ ) by

$$\begin{aligned}
\mathcal{F}_{\text{tail(tail)}}(u) &= \frac{4G^3 M^2}{5c^{11}} I_{ij}^{(3)}(u) \int_0^{+\infty} d\tau I_{ij}^{(6)}(u-\tau) \left[ \log^2 \left( \frac{c\tau}{2r} \right) + \frac{57}{70} \log \left( \frac{c\tau}{2r} \right) + \frac{124627}{44100} \right] \\
&- \frac{428G^3 M^2}{525c^{11}} \log \left( \frac{r}{r_0} \right) I_{ij}^{(3)}(u) I_{ij}^{(5)}(u), \tag{2.15a}
\end{aligned}$$

$$\mathcal{F}_{(\text{tail})^2}(u) = \frac{4G^3 M^2}{5c^{11}} \left( \int_0^{+\infty} d\tau I_{ij}^{(5)}(u-\tau) \left[ \log \left( \frac{c\tau}{2r} \right) + \frac{11}{12} \right] \right)^2. \tag{2.15b}$$

We shall come back to the discussion of the hereditary terms in detail in Sec. VI where we shall compute their contributions to far-zone energy flux.

### III. THE EQUATIONS OF MOTION AND THE CONSERVED ENERGY FOR HEAD-ON COLLISION

#### A. Standard Harmonic Coordinate System

By standard harmonic coordinates we refer to the coordinate system that has been used in previous works [18, 28]. Since the head-on collision problem has only one direction of motion one can write relevant equations for the head-on case by imposing the restrictions

$$\mathbf{x} = z \hat{\mathbf{n}}, \quad \mathbf{v} = \dot{z} \hat{\mathbf{n}}, \quad r = z, \quad v = \dot{r} = \dot{z}, \tag{3.1}$$

on the corresponding expression for general orbits in terms of  $z$  and  $\dot{z}$ , where  $z$  is the separation between the two objects at a given time and  $\dot{z}$  is the first time derivative of  $z$ , giving (coordinate) speed with which they are moving with respect to each other at that instant.

The computation of the energy flux involves time derivatives of source multipole moments which in turn will require the knowledge of equations of motion at appropriate PN order. Computing the 3PN accurate energy flux requires the

3PN accurate equations of motion [28]. 3PN accurate equations of motion in the CM frame associated with standard harmonic coordinate system, for compact objects moving in generic orbits, are given in [25, 28].

Since we will discuss the results in other coordinate systems like the modified harmonic coordinates and the ADM coordinates in the subsequent sections, we will provide for a more general discussion of the 2.5PN terms along the lines of [29] based on [30, 31] (see also [32]). In addition to the contact transformation involving “conservative” orders up to 3PN required to go from the SH to the modified harmonic and ADM coordinates (involving even order 2PN and 3PN terms) there still remains the possible change of gauge in the radiation reaction (dissipative) terms at order 2.5PN. Recall, that in the SH coordinate system the lowest-order dissipative part of the equations of motion, i.e. the 2.5PN acceleration term, is given by (with boldface letters indicating ordinary three-dimensional vectors)

$$\mathbf{a}_{2.5\text{PN}}^{\text{SH}} = \frac{8}{5} \frac{G^2 m^2 \nu}{c^5 r^3} \left( \left[ 3v^2 + \frac{17}{3} \frac{Gm}{r} \right] \dot{r} \mathbf{n} + \left[ -v^2 - 3 \frac{Gm}{r} \right] \mathbf{v} \right). \quad (3.2)$$

One may however prefer to employ alternative radiation gauges and a convenient characterization at 2.5PN order has been investigated earlier in [30, 31]. Following this work, the most general form of the 2.5PN term in the relative acceleration is specified by the two-parameter family written as,

$$\mathbf{a}_{2.5\text{PN}}^{\text{gen}} = \frac{8}{5} \frac{G^2 m^2 \nu}{c^5 r^3} \left( A_{2.5\text{PN}} \dot{r} \mathbf{n} + B_{2.5\text{PN}} \mathbf{v} \right), \quad (3.3a)$$

$$A_{2.5\text{PN}} \equiv 3(1 + \beta)v^2 + \frac{1}{3}(23 + 6\alpha - 9\beta) \frac{Gm}{r} - 5\beta \dot{r}^2, \quad (3.3b)$$

$$B_{2.5\text{PN}} \equiv -(2 + \alpha)v^2 - (2 - \alpha) \frac{Gm}{r} + 3(1 + \alpha)\dot{r}^2. \quad (3.3c)$$

The general 2.5PN gauge is parametrized by the two numerical constants  $\alpha$  and  $\beta$ . The SH (and modified harmonic) gauge in which the acceleration is given by (3.2) corresponds to the choice  $\alpha = -1$  and  $\beta = 0$ ; the ADM gauge corresponds to  $\alpha = 5/3$  and  $\beta = 3$ , in which case the 2.5PN acceleration becomes [33]

$$\mathbf{a}_{2.5\text{PN}}^{\text{ADM}} = \frac{8}{5} \frac{G^2 m^2 \nu}{c^5 r^3} \left( \left[ 12v^2 + 2 \frac{Gm}{r} - 15\dot{r}^2 \right] \dot{r} \mathbf{n} + \left[ -\frac{11}{3}v^2 - \frac{1}{3} \frac{Gm}{r} + 8\dot{r}^2 \right] \mathbf{v} \right). \quad (3.4)$$

By imposing the restrictions given by Eq. (3.1) we can write the equations of motion (or acceleration) in terms of the variables  $z$  and  $\dot{z}$  for head-on situation as  $a^i = a n^i$ , where,

$$\begin{aligned} a = & -\frac{Gm}{z^2} \left\{ 1 + \frac{1}{c^2} \left[ \dot{z}^2 \left( -3 + \frac{7}{2} \nu \right) + \frac{Gm}{z} (-4 - 2\nu) \right] \right. \\ & + \frac{1}{c^4} \left[ \dot{z}^4 \left( -\frac{21}{8} \nu - \frac{21}{8} \nu^2 \right) + \frac{Gm}{z} \dot{z}^2 (-11\nu + 4\nu^2) + \frac{G^2 m^2}{z^2} \left( 9 + \frac{87}{4} \nu \right) \right] \\ & + \frac{\dot{z}}{c^5} \left[ \frac{Gm}{z} \dot{z}^2 \left( -\frac{32}{5} - \frac{16}{5} \alpha + \frac{16}{5} \beta \right) \nu + \frac{G^2 m^2}{z^2} \left( -\frac{136}{15} - \frac{24}{5} \alpha + \frac{24}{5} \beta \right) \nu \right] \\ & + \frac{1}{c^6} \left[ \dot{z}^6 \left( -\frac{19}{16} \nu + \frac{13}{16} \nu^2 + \frac{147}{16} \nu^3 \right) \right. \\ & + \frac{Gm}{z} \dot{z}^4 \left( \frac{199}{12} \nu - 8\nu^2 - 12\nu^3 \right) \\ & + \frac{G^2 m^2}{z^2} \dot{z}^2 \left( -3 + \left[ \frac{117709}{840} + \frac{123}{32} \pi^2 - 44 \log \left( \frac{z}{z'_0} \right) \right] \nu + \frac{211}{8} \nu^2 + 2\nu^3 \right) \\ & \left. \left. + \frac{G^3 m^3}{z^3} \left( -16 + \left[ -\frac{437}{4} + \frac{41}{16} \pi^2 \right] \nu - \frac{71}{2} \nu^2 \right) \right] \right\} + \mathcal{O}(7). \end{aligned} \quad (3.5)$$

The general expression for center-of-mass energy  $E$  associated with standard harmonic coordinate system is given in [28] and the corresponding expression for head-on situation can be obtained by imposing restrictions given by Eq. (3.1). Thus we have

$$\frac{E_{\text{SH}}(z, \dot{z})}{\mu} = -\frac{Gm}{z} + \frac{\dot{z}^2}{2}$$



$$\begin{aligned}
& + \frac{1}{c^2} \left[ \dot{z}^4 \left( \frac{3}{8} - \frac{9}{8} \nu \right) + \frac{Gm}{z} \dot{z}^2 \left( \frac{3}{2} + \nu \right) + \frac{1}{2} \frac{G^2 m^2}{z^2} \right] \\
& + \frac{1}{c^4} \left[ \dot{z}^6 \left( \frac{5}{16} - \frac{35}{16} \nu + \frac{65}{16} \nu^2 \right) + \frac{Gm}{z} \dot{z}^4 \left( \frac{21}{8} - 3\nu - 6\nu^2 \right) + \frac{G^2 m^2}{z^2} \dot{z}^2 \left( \frac{9}{4} + \frac{7}{4} \nu + 2\nu^2 \right) \right. \\
& \quad \left. + \frac{G^3 m^3}{z^3} \left( -\frac{1}{2} - \frac{15}{4} \nu \right) \right] \\
& + \frac{1}{c^6} \left[ \dot{z}^8 \left( \frac{35}{128} - \frac{413}{128} \nu + \frac{833}{64} \nu^2 - \frac{2261}{128} \nu^3 \right) \right. \\
& \quad + \frac{Gm}{z} \dot{z}^6 \left( \frac{55}{16} - 15\nu + \frac{25}{4} \nu^2 + 35\nu^3 \right) \\
& \quad + \frac{G^2 m^2}{z^2} \dot{z}^4 \left( \frac{147}{16} - \frac{569}{48} \nu - \frac{245}{16} \nu^2 - 21\nu^3 \right) \\
& \quad + \frac{G^3 m^3}{z^3} \dot{z}^2 \left( \frac{11}{4} + \left[ -\frac{9719}{420} - \frac{41}{32} \pi^2 + \frac{44}{3} \log \left( \frac{z}{z'_0} \right) \right] \nu + \frac{15}{2} \nu^2 + 4\nu^3 \right) \\
& \quad \left. + \frac{G^4 m^4}{z^4} \left( \frac{3}{8} + \left[ \frac{18469}{840} - \frac{22}{3} \log \left( \frac{z}{z'_0} \right) \right] \nu \right) \right] + \mathcal{O}(8). \tag{3.6}
\end{aligned}$$

To study the head-on infall we consider two different situations, following [16]. In the first case (we will call it case I) we assume that the radial infall proceeds from rest at a finite initial separation whereas in the second case (case II), we assume the objects start falling towards each other from rest at infinite separation.

### 1. Case I: Infall from a finite distance

Let us suppose the two objects initially separated by the distance  $z_i$  start falling radially towards each other from the rest *i.e.*  $\dot{z}(z_i) = 0$ . Hence the center-of-mass energy  $E$  in standard harmonic coordinates at  $z_i$  will be

$$E_{\text{SH}}(z_i, 0) = -\mu c^2 \gamma_i \left\{ 1 - \frac{1}{2} \gamma_i + \left[ \frac{1}{2} + \frac{15}{4} \nu \right] \gamma_i^2 + \left[ -\frac{3}{8} + \left( -\frac{18469}{840} + \frac{22}{3} \log \left( \frac{z_i}{z'_0} \right) \right) \nu \right] \gamma_i^3 \right\}, \tag{3.7}$$

where  $\gamma_i = Gm/z_i c^2$ . Equating, Eqs. (3.6) and (3.7), the resultant expression can be inverted for  $\dot{z}$ .

$$\begin{aligned}
\dot{z}(z, z_i) = & -\sqrt{2} c \frac{\sqrt{\gamma}}{\sqrt{1-s}} \left\{ 1 - s + \left[ -\frac{5}{2} + \frac{5}{4} \nu + s \left( 3 - \frac{7}{2} \nu \right) + s^2 \left( -\frac{1}{2} + \frac{9}{4} \nu \right) \right] \gamma \right. \\
& + \left[ \frac{27}{8} - 7\nu + \frac{55}{32} \nu^2 + s \left( -\frac{37}{8} + \frac{179}{8} \nu - \frac{173}{32} \nu^2 \right) + s^2 \left( \frac{13}{8} - \frac{119}{8} \nu + \frac{165}{32} \nu^2 \right) + s^3 \left( -\frac{3}{8} - \frac{1}{2} \nu - \frac{47}{32} \nu^2 \right) \right] \gamma^2 \\
& + \left[ -\frac{65}{16} + \left[ \frac{64343}{1120} + \frac{41}{32} \pi^2 - 11 \log \left( \frac{z}{z'_0} \right) \right] \nu - \frac{945}{64} \nu^2 + \frac{237}{128} \nu^3 \right. \\
& + s \left( \frac{23}{4} + \left[ -\frac{73567}{560} - \frac{41}{32} \pi^2 + \frac{44}{3} \log \left( \frac{z}{z'_0} \right) \right] \nu + \frac{1953}{32} \nu^2 - \frac{63}{8} \nu^3 \right) \\
& + s^2 \left( -\frac{21}{8} + \frac{3271}{48} \nu - \frac{2507}{32} \nu^2 + \frac{633}{64} \nu^3 \right) + s^3 \left( \frac{5}{4} - \frac{43}{16} \nu + \frac{729}{32} \nu^2 - \frac{65}{16} \nu^3 \right) \\
& \left. + s^4 \left( -\frac{5}{16} + \left[ \frac{28433}{3360} - \frac{11}{3} \log \left( \frac{z_i}{z'_0} \right) \right] \nu + \frac{595}{64} \nu^2 + \frac{25}{128} \nu^3 \right) \right] \gamma^3 \right\}, \tag{3.8}
\end{aligned}$$

where  $\gamma = Gm/z c^2$  is the PN parameter and  $s = z/z_i < 1$ .

### 2. Case II: Infall from infinity

We can view case II as a limiting case of case I and the expression for  $\dot{z}$  can be obtained by inserting  $s = z/z_i$  back in Eq. (3.8) and taking the limit when  $z_i \rightarrow \infty$ . For  $\dot{z}$  in SH coordinates, we have

$$\dot{z}(z) = -\sqrt{2} c \sqrt{\gamma} \left\{ 1 + \left[ -\frac{5}{2} + \frac{5}{4} \nu \right] \gamma + \left[ \frac{27}{8} - 7\nu + \frac{55}{32} \nu^2 \right] \gamma^2 \right.$$

$$+ \left[ -\frac{65}{16} + \left[ \frac{64343}{1120} + \frac{41}{32} \pi^2 - 11 \log \left( \frac{z}{z'_0} \right) \right] \nu - \frac{945}{64} \nu^2 + \frac{237}{128} \nu^3 \right] \gamma^3 \Big\} . \quad (3.9)$$

As expected, for  $\nu = 0$  the above relation is consistent with the radial geodesics of Schwarzschild in *standard harmonic coordinates* [16] .

### B. Modified Harmonic Coordinate System

The SH coordinates are useful for analytical algebraic checks but also contain some gauge-dependent logarithms which are less suitable for numerical computations. It has been shown in [25] that such dependences can be transformed away by using suitable gauge transformations. The expression for the shift “ $\delta_{(\text{SH} \rightarrow \text{MH})} E$ ” for general orbit case has been given by Eq. (4.12) of [25] and we have used Eq. (3.1) to obtain the corresponding expression for head-on situation. We can write the center-of-mass energy in MH coordinates using the relation

$$E_{\text{MH}} = E_{\text{SH}} + \delta_{(\text{SH} \rightarrow \text{MH})} E , \quad (3.10)$$

where  $E_{\text{SH}}$  represents the energy  $E$  in SH coordinates and is given by Eq. (3.6). “ $\delta_{(\text{MH} \rightarrow \text{SH})} E$ ” for head-on case reads

$$\delta_{(\text{SH} \rightarrow \text{MH})} E = \frac{22}{3} \frac{G^3 m^4 \nu^2}{c^6 z^3} \left\{ \left[ \frac{G m}{z} - 2 \dot{z}^2 \right] \log \left( \frac{z}{z'_0} \right) + \dot{z}^2 \right\} + \mathcal{O}(8) . \quad (3.11)$$

#### 1. Case I: Infall from a finite distance

It is evident from the above that using Eqs. (3.6) and (3.11) in Eq. (3.10) we can write the expression for conserved energy  $E$  in MH coordinates. At the initial separation  $z_i$  energy in MH coordinates reads

$$E_{\text{MH}}(z = z_i) = -\mu c^2 \gamma_i \left\{ 1 - \frac{1}{2} \gamma_i + \left[ \frac{1}{2} + \frac{15}{4} \nu \right] \gamma_i^2 + \left[ -\frac{3}{8} - \frac{18469}{840} \nu \right] \gamma_i^3 \right\} . \quad (3.12)$$

By equating Eq. (3.12) and the expression for center-of-mass energy in MH coordinates and then inverting the resultant expression, one can obtain the expression for  $\dot{z}$  in MH coordinates. For brevity in presentation, in what follows, we will list only the differences in various expressions in a particular coordinate system from their SH values. By adding the difference to the SH expression the corresponding expression in the relevant coordinate can be computed. In particular in this case, for  $\dot{z}$  we have,

$$\dot{z}_{\text{MH}} = \dot{z}_{\text{SH}} + \delta_{(\text{SH} \rightarrow \text{MH})} \dot{z} , \quad (3.13)$$

where,

$$\begin{aligned} \delta_{(\text{SH} \rightarrow \text{MH})} \dot{z} = & -\sqrt{2} c \frac{\gamma^{7/2}}{\sqrt{1-s}} \left\{ \left( -\frac{22}{3} + 11 \log \left( \frac{z}{z'_0} \right) \right) \nu + s \left( \frac{22}{3} - \frac{44}{3} \log \left( \frac{z}{z'_0} \right) \right) \nu \right. \\ & \left. + s^4 \left( \frac{11}{3} \log \left( \frac{z_i}{z'_0} \right) \right) \nu \right\} . \end{aligned} \quad (3.14)$$

Though to avoid heavy notation we write  $z$  and  $\dot{z}$ , beware that in this subsection they correspond to  $z_{\text{MH}}$  and  $\dot{z}_{\text{MH}}$  respectively and in the next subsection to  $z_{\text{ADM}}$  and  $\dot{z}_{\text{ADM}}$  respectively.

#### 2. Case II: Infall from infinity

Once again, by inserting  $s = z/z_i$  back in Eq. (3.14) and taking the limit when  $z_i \rightarrow \infty$  we obtain the expression for “ $\delta_{(\text{SH} \rightarrow \text{MH})} \dot{z}$ ” as

$$\delta_{(\text{SH} \rightarrow \text{MH})} \dot{z} = -\sqrt{2} c \gamma^{7/2} \left[ -\frac{22}{3} + 11 \log \left( \frac{z}{z'_0} \right) \right] \nu . \quad (3.15)$$



### C. ADM coordinate System

Finally, in this section we provide the expressions for the conserved energy in ADM coordinates. Like MH coordinate systems the ADM coordinate system is also free from logarithms appearing in 3PN expressions of EOM or source multipole moments when standard harmonic coordinate system is used. We can write the center-of-mass energy in ADM coordinates using the relation

$$E_{\text{ADM}} = E_{\text{SH}} + \delta_{(\text{SH} \rightarrow \text{ADM})} E, \quad (3.16)$$

where  $E_{\text{SH}}$  is given by Eq. (3.6) and for “ $\delta_{(\text{SH} \rightarrow \text{ADM})} E$ ” in head-on situation (see Appendix A for its computation) we have

$$\begin{aligned} \delta_{(\text{SH} \rightarrow \text{ADM})} E = & \frac{G^2 m^3 \nu}{c^4 z^2} \left\{ \dot{z}^2 \left( -\frac{1}{4} - \frac{5}{4} \nu \right) + \frac{G m}{z} \left( \frac{1}{4} + 3 \nu \right) \right\} \\ & + \frac{G^2 m^3 \nu}{c^6 z^2} \left\{ \dot{z}^4 \left( -\frac{3}{8} + \frac{20}{3} \nu + 4 \nu^2 \right) \right. \\ & + \frac{G m}{z} \dot{z}^2 \left( -\frac{9}{8} + \left[ \frac{5441}{280} + \frac{21}{16} \pi^2 - \frac{44}{3} \log \left( \frac{z}{z'_0} \right) \right] \nu - \frac{55}{8} \nu^2 \right) \\ & \left. + \frac{G^2 m^2}{z^2} \left( -\frac{1}{4} + \left[ -\frac{3613}{280} - \frac{21}{32} \pi^2 + \frac{22}{3} \log \left( \frac{z}{z'_0} \right) \right] \nu \right) \right\}. \end{aligned} \quad (3.17)$$

#### 1. Case I: Infall from a finite distance

$$E_{\text{ADM}}(z = z_i) = -\mu c^2 \gamma_i \left\{ 1 - \frac{1}{2} \gamma_i + \left[ \frac{1}{4} + \frac{3}{4} \nu \right] \gamma_i^2 + \left[ -\frac{1}{8} + \left( -\frac{109}{12} + \frac{21}{32} \pi^2 \right) \nu \right] \gamma_i^3 \right\}. \quad (3.18)$$

For  $\dot{z}$  in ADM coordinate we have,

$$\dot{z}_{\text{ADM}} = \dot{z}_{\text{SH}} + \delta_{(\text{SH} \rightarrow \text{ADM})} \dot{z}, \quad (3.19)$$

where,

$$\begin{aligned} \delta_{(\text{SH} \rightarrow \text{ADM})} \dot{z} = & -\sqrt{2} c \frac{\sqrt{\gamma}}{\sqrt{1-s}} \left\{ \left[ \frac{1}{8} - \frac{\nu}{4} + s \left( -\frac{1}{4} - \frac{5}{4} \nu \right) + s^3 \left( \frac{1}{8} + \frac{3}{2} \nu \right) \right] \gamma^2 \right. \\ & + \left[ \frac{5}{16} + \left( -\frac{101959}{3360} - \frac{63}{64} \pi^2 + 11 \log \left( \frac{z}{z'_0} \right) \right) \nu + \frac{9}{16} \nu^2 \right. \\ & + s \left( \frac{1}{16} + \left[ \frac{179147}{3360} + \frac{21}{16} \pi^2 - \frac{44}{3} \log \left( \frac{z}{z'_0} \right) \right] \nu - \frac{41}{8} \nu^2 \right) \\ & + s^2 \left( -\frac{1}{8} - \frac{715}{48} \nu + \frac{97}{16} \nu^2 \right) + s^3 \left( -\frac{7}{16} - \frac{145}{32} \nu + \frac{69}{8} \nu^2 \right) \\ & \left. \left. + s^4 \left( \frac{3}{16} + \left[ -\frac{3971}{1120} - \frac{21}{64} \pi^2 + \frac{11}{3} \log \left( \frac{z_i}{z'_0} \right) \right] \nu - \frac{81}{8} \nu^2 \right) \right] \gamma^3 \right\}. \end{aligned} \quad (3.20)$$

#### 2. Case II: Infall from infinity

The expression for “ $\delta_{(\text{SH} \rightarrow \text{ADM})} \dot{z}$ ” can be written by inserting  $s = z/z_i$  back in Eq. (3.20) and taking the limit when  $z_i \rightarrow \infty$  as

$$\begin{aligned} \delta_{(\text{SH} \rightarrow \text{ADM})} \dot{z} = & \sqrt{2} c \sqrt{\gamma} \left\{ \left( \frac{1}{8} - \frac{\nu}{4} \right) \gamma^2 \right. \\ & \left. + \left( \left[ \frac{5}{16} + \left( -\frac{101959}{3360} - \frac{63}{64} \pi^2 + 11 \log \left( \frac{z}{z'_0} \right) \right) \nu + \frac{9}{16} \nu^2 \right] \right) \gamma^3 \right\}. \end{aligned} \quad (3.21)$$

### D. Inputs for the computation of the hereditary part

It is evident from Eqs. (2.14)-(2.15) that all integrals need to be evaluated with just Newtonian order accuracy except the one in the first term of Eq. (2.14) which needs to be computed with 1PN accuracy and hence in this section we provide 1PN accurate inputs which will be required for the computation of hereditary part of the energy flux. Since at 1PN order the expressions for all desired inputs [*e.g.* source moments, trajectory of the problem and relation connecting ADM mass to total mass ( $m = m_1 + m_2$ )] are the same in all the three coordinate systems we need not give these inputs in different coordinate systems.

#### 1. Case I: Infall from a finite distance

Equation (3.8) gives the expression for  $\dot{z}$  which in the 1PN limit can be expressed as

$$\dot{z} = -\sqrt{\frac{2Gm}{z}} \sqrt{1-s} \left[ 1 + \frac{Gm}{c^2 z} \left( -\frac{5}{2} + \frac{5}{4}\nu + s \left( \frac{1}{2} - \frac{9}{4}\nu \right) \right) \right]. \quad (3.22)$$

Solving Eq. (3.22) we get the 1PN trajectory as

$$t = \frac{z_i^{3/2}}{\sqrt{2Gm}} \left[ g(s) - \frac{1}{2} \frac{Gm}{c^2 z_i} h_0(s) - \nu \frac{1}{2} \frac{Gm}{c^2 z_i} h_1(s) \right], \quad (3.23)$$

where  $g(s)$ ,  $h_0(s)$  and  $h_1(s)$  are the following simple linear combinations of the elementary functions  $f_1(s) = \sqrt{s}\sqrt{1-s}$  and  $f_2(s) = \arcsin \sqrt{s}$ ,

$$g(s) = f_1(s) - f_2(s), \quad (3.24a)$$

$$h_0(s) = f_1(s) + 9f_2(s), \quad (3.24b)$$

$$h_1(s) = -\frac{1}{2}(9f_1(s) + f_2(s)). \quad (3.24c)$$

#### 2. Case II: Infall from infinity

In the 1PN limit Eq. (3.9) gives

$$\dot{z} = -\sqrt{\frac{2Gm}{z}} \left[ 1 - \frac{5}{2} \frac{Gm}{c^2 z} \left( 1 - \frac{\nu}{2} \right) \right]. \quad (3.25)$$

By integrating Eq. (3.25) we get the 1PN trajectory of the problem which reads

$$t = -\frac{\sqrt{2} z^{3/2}}{3\sqrt{G}\sqrt{m}} \left[ 1 + \frac{15}{2} \frac{Gm}{c^2 z} \left( 1 - \frac{\nu}{2} \right) \right]. \quad (3.26)$$

## IV. THE MULTIPOLE MOMENTS OF COMPACT BINARY SYSTEMS

In this section we shall provide the expressions for source multipole moments with an accuracy sufficient for the computation of the 3PN accurate energy flux in standard harmonic coordinates. General expressions for these moments have been given in [25] for inspiralling compact objects moving in generic orbits in standard harmonic coordinates. Since the head-on collision problem has only one direction of motion one can write the expressions for source moments for head-on case by imposing the restrictions, Eq. (3.1) on the corresponding expression for general orbits in terms of  $z$  and  $\dot{z}$ . Further, as discussed in [29], the 2.5PN term for general orbits in SH coordinates (see Eq. (3.1) of [25]) is modified by

$$\delta_\epsilon I_{ij} = -\frac{16}{15} \frac{G^2 m^3 \nu^2}{c^5} \left[ -\beta \dot{r} n_{\langle i} n_{j\rangle} + (3 + 3\alpha - 2\beta) n_{\langle i} v_{j\rangle} \right], \quad (4.1)$$

which for the head-on case reduces to  $-\frac{16}{5}\frac{G^2 m^3 \nu^2}{c^5} \dot{z}(1+\alpha-\beta) n_{\langle i} n_{j \rangle}$ . Thus, the 3PN mass quadrupole  $I_{ij}$  for head-on case reads as

$$\begin{aligned}
I_{ij} = & m z^2 \nu \left\{ 1 + \frac{1}{c^2} \left[ \dot{z}^2 \left( \frac{9}{14} - \frac{27}{14} \nu \right) + \frac{Gm}{z} \left( -\frac{5}{7} + \frac{8}{7} \nu \right) \right] \right. \\
& + \frac{1}{c^4} \left[ \dot{z}^4 \left( \frac{83}{168} - \frac{589}{168} \nu + \frac{1111}{168} \nu^2 \right) \right. \\
& + \frac{Gm}{z} \dot{z}^2 \left( \frac{32}{9} + \frac{289}{126} \nu - \frac{1195}{126} \nu^2 \right) \\
& + \left. \frac{G^2 m^2}{z^2} \left( -\frac{355}{252} - \frac{953}{126} \nu + \frac{337}{252} \nu^2 \right) \right] \\
& + \frac{\dot{z}}{c^5} \left[ \frac{G^2 m^2}{z^2} \left( \frac{8}{35} - \frac{16}{5} \alpha + \frac{16}{5} \beta \right) \nu \right] \\
& + \frac{1}{c^6} \left[ \dot{z}^6 \left( \frac{507}{1232} - \frac{6101}{1232} \nu + \frac{12525}{616} \nu^2 - \frac{34525}{1232} \nu^3 \right) \right. \\
& + \frac{Gm}{z} \dot{z}^4 \left( \frac{26177}{5544} - \frac{9313}{693} \nu - \frac{34165}{2772} \nu^2 + \frac{75787}{1386} \nu^3 \right) \\
& + \frac{G^2 m^2}{z^2} \dot{z}^2 \left( -\frac{8281261}{415800} + \frac{428}{105} \log \left( \frac{z}{z_0} \right) + \frac{15703}{792} \nu + \frac{13803}{616} \nu^2 - \frac{527797}{16632} \nu^3 \right) \\
& + \frac{G^3 m^3}{z^3} \left( \frac{6285233}{207900} - \frac{428}{105} \log \left( \frac{z}{z_0} \right) + \left[ \frac{15502}{385} - \frac{44}{3} \log \left( \frac{z}{z'_0} \right) \right] \nu - \frac{3632}{693} \nu^2 \right. \\
& + \left. \left. \frac{13289}{8316} \nu^3 \right) \right] \left. \right\} n_{\langle i} n_{j \rangle} + \mathcal{O}(7). \tag{4.2}
\end{aligned}$$

Note that the quantity  $z_0$  appearing in above expression is, in our present head-on notation, the constant length scale  $r_0$  appearing in Eq. (2.6) and in the relations connecting radiative multipole moments and source multipole moments. The presence of the other constant  $z'_0$  through some logarithms  $\log(z/z'_0)$  at 3PN order is due to the use of standard harmonic coordinates and corresponds to  $r'_0$  in the earlier papers. Later in this paper we shall show that alternatively one can use other coordinate systems such as the MH or ADM coordinate system which do not involve such logarithms.

The 2PN mass octupole  $I_{ijk}$  for head-on case is given by

$$\begin{aligned}
I_{ijk} = & -m z^3 \sqrt{1-4\nu} \nu \left\{ 1 + \frac{1}{c^2} \left[ \dot{z}^2 \left( \frac{5}{6} - \frac{19}{6} \nu \right) + \frac{Gm}{z} \left( -\frac{5}{6} + \frac{13}{6} \nu \right) \right] \right. \\
& + \frac{1}{c^4} \left[ \dot{z}^4 \left( \frac{61}{88} - \frac{1579}{264} \nu + \frac{1129}{88} \nu^2 \right) \right. \\
& + \frac{Gm}{z} \dot{z}^2 \left( \frac{54}{11} + \frac{521}{132} \nu - \frac{2467}{132} \nu^2 \right) \\
& + \left. \frac{G^2 m^2}{z^2} \left( -\frac{47}{33} - \frac{1591}{132} \nu + \frac{235}{66} \nu^2 \right) \right] \left. \right\} n_{\langle i} n_j n_{k \rangle} + \mathcal{O}(5). \tag{4.3}
\end{aligned}$$

The 1PN mass moment,  $I_{ijkl}$  reads as

$$\begin{aligned}
I_{ijkl} = & m z^4 \nu \left\{ 1 - 3\nu + \frac{1}{c^2} \left[ \dot{z}^2 \left( \frac{23}{22} - \frac{159}{22} \nu + \frac{291}{22} \nu^2 \right) \right. \right. \\
& + \left. \left. \frac{Gm}{z} \left( -\frac{10}{11} + \frac{61}{11} \nu - \frac{105}{11} \nu^2 \right) \right] \right\} n_{\langle i} n_j n_k n_{l \rangle} + \mathcal{O}(4). \tag{4.4}
\end{aligned}$$

The moment,  $I_{ijklm}$  which will be needed with Newtonian accuracy is

$$I_{ijklm} = -m z^5 \sqrt{1-4\nu} (1-2\nu) \nu n_{\langle i} n_j n_k n_l n_{m \rangle} + \mathcal{O}(2). \tag{4.5}$$

Finally we give the monopole moment  $W$ , which is

$$W = \frac{1}{3} \nu m z \dot{z} + \mathcal{O}(2). \quad (4.6)$$

## V. INSTANTANEOUS CONTRIBUTIONS IN THE ENERGY FLUX FOR HEAD-ON INFALL

### A. The 3PN instantaneous energy flux in Standard Harmonic Coordinates

Having source multipole moments given by Eqs. (4.2)-(4.6) and equations of motion given by Eq. (3.5) with the desired PN accuracies one can compute the required time derivatives of source moments to get instantaneous contribution to the far-zone GW energy flux using Eq. (2.8). Since the instantaneous contribution to 3PN far-zone energy flux for compact binaries moving in general orbits has already been listed in [25] in terms of SH variables  $r$ ,  $\dot{r}$  and  $v$ , we can also directly write down the corresponding expression for instantaneous energy flux in terms of the variables  $z$  and  $\dot{z}$  for the head-on situation using Eq. (3.1). The form of the 2.5PN terms in the flux for a general 2.5PN gauge is discussed in [29] [see Eq.(3.14a) there], and we adapt it for the head-on case. We write the result as,

$$\mathcal{F}_{\text{inst}} = \mathcal{F}_{\text{inst}}^{\text{N}} + \mathcal{F}_{\text{inst}}^{\text{1PN}} + \mathcal{F}_{\text{inst}}^{\text{2PN}} + \mathcal{F}_{\text{inst}}^{\text{2.5PN}} + \mathcal{F}_{\text{inst}}^{\text{3PN}} + \mathcal{O}(7), \quad (5.1)$$

and find that the various PN pieces for the head-on case are given by

$$\mathcal{F}_{\text{inst}}^{\text{N}} = \frac{8}{15} \frac{G^3 m^4 \nu^2}{c^5 z^4} \dot{z}^2, \quad (5.2a)$$

$$\mathcal{F}_{\text{inst}}^{\text{1PN}} = \frac{8}{15} \frac{G^3 m^4 \nu^2}{c^7 z^4} \left\{ \dot{z}^4 \left( -\frac{32}{7} + \frac{18}{7} \nu \right) + \frac{Gm}{z} \dot{z}^2 \left( \frac{54}{7} + \frac{10}{7} \nu \right) + \frac{G^2 m^2}{z^2} \left( \frac{4}{7} - \frac{16}{7} \nu \right) \right\}, \quad (5.2b)$$

$$\begin{aligned} \mathcal{F}_{\text{inst}}^{\text{2PN}} = & \frac{8}{15} \frac{G^3 m^4 \nu^2}{c^9 z^4} \left\{ \dot{z}^6 \left( \frac{44}{21} - \frac{25}{21} \nu - \frac{67}{21} \nu^2 \right) + \frac{Gm}{z} \dot{z}^4 \left( -\frac{655}{21} + \frac{65}{63} \nu + \frac{1016}{63} \nu^2 \right) \right. \\ & \left. + \frac{G^2 m^2}{z^2} \dot{z}^2 \left( -\frac{9371}{189} + \frac{971}{14} \nu - \frac{140}{9} \nu^2 \right) + \frac{G^3 m^3}{z^3} \left( -\frac{506}{63} + \frac{228}{7} \nu - \frac{16}{9} \nu^2 \right) \right\}, \end{aligned} \quad (5.2c)$$

$$\mathcal{F}_{\text{inst}}^{\text{2.5PN}} = \frac{8}{15} \frac{G^3 m^4 \nu^2}{c^{10} z^4} \dot{z} \left\{ \frac{8}{7} \frac{Gm}{z} \dot{z}^4 + \frac{G^2 m^2}{z^2} \dot{z}^2 \left( \frac{96}{7} + \frac{48}{5} \alpha - \frac{48}{5} \beta \right) + \frac{G^3 m^3}{z^3} \left( \frac{332}{105} + \frac{16}{5} \alpha - \frac{16}{5} \beta \right) \right\} \nu, \quad (5.2d)$$

$$\begin{aligned} \mathcal{F}_{\text{inst}}^{\text{3PN}} = & \frac{8}{15} \frac{G^3 m^4 \nu^2}{c^{11} z^4} \left\{ \dot{z}^8 \left( \frac{360}{77} - \frac{3533}{924} \nu - \frac{247}{462} \nu^2 + \frac{806}{77} \nu^3 \right) \right. \\ & + \frac{Gm}{z} \dot{z}^6 \left( -\frac{863}{165} + \frac{24842}{693} \nu - \frac{33515}{462} \nu^2 - \frac{65705}{2772} \nu^3 \right) \\ & + \frac{G^2 m^2}{z^2} \dot{z}^4 \left( -\frac{840109}{9450} + \frac{856}{35} \log \left( \frac{z}{z_0} \right) \right) \\ & - \left[ \frac{234205}{396} + \frac{123}{16} \pi^2 \right] \nu + \frac{418973}{1386} \nu^2 + \frac{176927}{4158} \nu^3 \\ & + \frac{G^3 m^3}{z^3} \dot{z}^2 \left( -\frac{11963741}{51975} + \frac{3424}{105} \log \left( \frac{z}{z_0} \right) \right) \\ & - \left[ \frac{22682}{315} + \frac{205}{8} \pi^2 - \frac{176}{3} \log \left( \frac{z}{z_0} \right) \right] \nu + \frac{72477}{154} \nu^2 - \frac{20147}{231} \nu^3 \\ & \left. + \frac{G^4 m^4}{z^4} \left( \frac{37571}{693} - \frac{59848}{297} \nu - \frac{6038}{99} \nu^2 - \frac{3464}{2079} \nu^3 \right) \right\}. \end{aligned} \quad (5.2e)$$

The dependence of the result (5.1)–(5.2) on  $z'_0$  is due to our use of the SH coordinate system. We will transform away this dependence by making use of a different coordinate system such as MH coordinate system. The presence of constant  $z_0$  is not surprising as it was present in the expression of the mass quadrupole moment and hence appears in the final expression for the instantaneous part of the 3PN energy flux. This dependence of the instantaneous terms on the constant  $z_0$  should exactly cancel a similar contribution coming from the tail terms. We explicitly show this cancellation in the next section.

The general expression for the energy flux above for the head-on case takes a simpler form for radial infall from rest. In this case the velocity  $\dot{z}$  can be expressed solely in terms of the coordinate  $z$  and the initial coordinate separation

where it is at rest as shown in Sec. III. However, since we are working up to 3PN there is one last element to be taken into account before we can proceed. This relates to the infall velocity due to leading gravitational radiation reaction that induces a  $\dot{z}_{\text{RR}}$  at 2.5PN i.e.  $\dot{z}_{2.5\text{PN}}$ . To evaluate this we can adapt the treatment in [34] to the head-on case. It requires only the leading term in  $E$  and the GW energy flux  $\mathcal{F}$  and the infall due to radiation reaction for the finite separation case is given by

$$\dot{z}_{\text{RR}} = \frac{dE/dt}{dE/dz} = -\frac{\mathcal{F}}{dE/dz} = -\frac{16 G^3 m^3}{15 c^5 z^3} \nu(1-s). \quad (5.3)$$

Adding  $\dot{z}_{\text{RR}}$  to the  $\dot{z}$  given by Eq. (3.8) yields the complete 3PN accurate  $\dot{z}$  that will be employed in the next subsection to rewrite the energy flux solely in terms of the variable  $z$ . It should be obvious that the form of  $\dot{z}_{\text{RR}}$  is the same in all the three coordinate systems that we use in this paper.

We now have all basic ingredients for the computation of the 3PN GW energy flux from compact objects with arbitrary mass ratios falling radially towards each other and can proceed to compute the instantaneous part of the energy flux for the head-on infall case.

### 1. Case I: Infall from a finite distance

Starting from the 3PN instantaneous contribution to energy flux in SH coordinates [Eq. (5.1)-(5.2)] in terms of the variables  $z$  and  $\dot{z}$  and substituting the expression for  $\dot{z}$  given by Eq. (3.8) supplemented by  $\dot{z}_{\text{RR}}$  given by Eq. (5.3) we get the final expression for energy flux in terms of just one variable  $z$ . The final expression for the instantaneous part of energy flux in standard harmonic coordinate reads as

$$\begin{aligned} \left(\frac{d\mathcal{E}}{dt}\right)_{\text{SH}}^{\text{inst}} = & \frac{16}{15} \frac{c^5}{G} \nu^2 \gamma^5 \left\{ 1 - s + \left[ -\frac{43}{7} + \frac{111}{14} \nu + s \left( \frac{116}{7} - \frac{131}{7} \nu \right) + s^2 \left( -\frac{71}{7} + \frac{135}{14} \nu \right) \right] \gamma \right. \\ & + \left[ -\frac{1127}{27} - \frac{803}{36} \nu + \frac{112}{3} \nu^2 + s \left( -\frac{4471}{189} + \frac{15481}{63} \nu - \frac{2864}{21} \nu^2 \right) \right. \\ & + s^2 \left( \frac{1870}{21} - \frac{5503}{18} \nu + \frac{8800}{63} \nu^2 \right) + s^3 \left( -\frac{83}{3} + \frac{1183}{12} \nu - \frac{872}{21} \nu^2 \right) \left. \right] \gamma^2 \\ & + \sqrt{2} \sqrt{1-s} \left[ \left( -\frac{578}{35} - \frac{56}{5} \alpha + \frac{56}{5} \beta \right) \nu + s \left( \frac{1808}{105} + \frac{48}{5} \alpha - \frac{48}{5} \beta \right) \nu - \frac{16}{7} s^2 \nu \right] \gamma^{5/2} \\ & + \left[ -\frac{195203}{20790} + \frac{1712}{21} \log\left(\frac{z}{z_0}\right) + \left[ -\frac{2855087}{2772} - \frac{615}{16} \pi^2 + \frac{110}{3} \log\left(\frac{z}{z'_0}\right) \right] \nu + \frac{1738129}{5544} \nu^2 + \frac{231520}{2079} \nu^3 \right. \\ & + s \left( \frac{4260593}{17325} - \frac{13696}{105} \log\left(\frac{z}{z_0}\right) + \left[ \frac{23954477}{41580} + \frac{861}{16} \pi^2 - \frac{88}{3} \log\left(\frac{z}{z'_0}\right) \right] \nu + \frac{3228377}{2772} \nu^2 - \frac{1366537}{2079} \nu^3 \right) \\ & + s^2 \left( -\frac{19591864}{51975} + \frac{1712}{35} \log\left(\frac{z}{z_0}\right) + \left[ \frac{523423}{308} - \frac{123}{8} \pi^2 \right] \nu - \frac{47223}{14} \nu^2 + \frac{304961}{297} \nu^3 \right) \\ & + s^3 \left( \frac{212164}{1155} - \frac{2192423}{1386} \nu + \frac{897295}{396} \nu^2 - \frac{406498}{693} \nu^3 \right) \\ & + s^4 \left( -\frac{1245}{77} + \left[ \frac{1088117}{4620} - \frac{22}{3} \log\left(\frac{z_i}{z'_0}\right) \right] \nu - \frac{105989}{264} \nu^2 + \frac{8076}{77} \nu^3 \right) \left. \right] \gamma^3 \left. \right\}. \quad (5.4) \end{aligned}$$

The standard harmonic gauge at 2.5PN corresponds to  $\alpha = -1$  and  $\beta = 0$ .

### 2. Case II: Infall from infinity

As discussed in the previous section, we can see case II as a special case of case I and the expression  $d\mathcal{E}/dt$  can be obtained by inserting  $s = z/z_i$  in Eq. (5.4) and taking the limit when  $z_i \rightarrow \infty$ . The instantaneous part of energy flux in SH coordinates reads,

$$\left(\frac{d\mathcal{E}}{dt}\right)_{\text{SH}}^{\text{inst}} = \frac{16}{15} \frac{c^5}{G} \nu^2 \gamma^5 \left\{ 1 + \left[ -\frac{43}{7} + \frac{111}{14} \nu \right] \gamma + \left[ -\frac{1127}{27} - \frac{803}{36} \nu + \frac{112}{3} \nu^2 \right] \gamma^2 + \sqrt{2} \left[ \left( -\frac{578}{35} - \frac{56}{5} \alpha + \frac{56}{5} \beta \right) \nu \right] \gamma^{5/2} \right\}$$

$$+ \left[ -\frac{195203}{20790} + \frac{1712}{21} \log \left( \frac{z}{z_0} \right) + \left[ -\frac{2855087}{2772} - \frac{615}{16} \pi^2 + \frac{110}{3} \log \left( \frac{z}{z'_0} \right) \right] \nu + \frac{1738129}{5544} \nu^2 + \frac{231520}{2079} \nu^3 \right] \gamma^3 \Big\} . \quad (5.5)$$

The standard harmonic gauge at 2.5PN corresponds to  $\alpha = -1$  and  $\beta = 0$ .

### B. The 3PN instantaneous Energy Flux in Modified Harmonic Coordinates

As we have pointed out in Sec. IIIB that one needs to use an alternative coordinate system such as MH coordinate system, which is more suitable for numerical computations. In this section we provide the 3PN energy flux expressions in MH coordinates. We can write the energy flux  $\mathcal{F}$  in the MH coordinates by using the relation

$$\mathcal{F}_{\text{MH}} = \mathcal{F}_{\text{SH}} + \delta_{(\text{SH} \rightarrow \text{MH})} \mathcal{F}, \quad (5.6)$$

where  $\mathcal{F}_{\text{SH}}$  is the energy flux in SH coordinates for head-on situation given by Eqs. (5.1)-(5.2). The general expression for the shift “ $\delta_{(\text{SH} \rightarrow \text{MH})} \mathcal{F}$ ” is given by Eq. (6.8) of [25], which in the head-on situation reduces to the following form when we use the restrictions given by Eq. (3.1),

$$\delta_{(\text{SH} \rightarrow \text{MH})} \mathcal{F} = -\frac{1408}{15} \frac{G^6 m^7 \nu^3}{c^{11} z^7} \left[ \frac{1}{3} \dot{z}^2 \log \left( \frac{z}{z'_0} \right) - \frac{\dot{z}^2}{12} + \mathcal{O}(2) \right]. \quad (5.7)$$

Once we have expressions for the energy flux  $\mathcal{F}$  in MH coordinates, we can compute the final expression for instantaneous part of energy flux in MH coordinates following the procedure adopted in Sec. VA.

#### 1. Case I: Infall from a finite distance

Substituting for  $\dot{z}$  in MH coordinates, in the expression for energy flux in MH coordinates, one can obtain the expression for the instantaneous part of far-zone radiative energy flux in MH coordinates as a function of the separation between the two objects at some instant. Rather than writing the full expression for energy flux in MH coordinates we list here the difference to be added to the expression in SH coordinates [Eq. (5.4)] to obtain the corresponding expression in MH coordinates. We have,

$$\delta_{(\text{SH} \rightarrow \text{MH})} (d\mathcal{E}/dt)_{\text{inst}} = \frac{16 c^5}{15 G} \nu^2 \gamma^8 \left[ -\frac{110}{3} \nu \log \left( \frac{z}{z'_0} \right) + s \left( \frac{88}{3} \nu \log \left( \frac{z}{z'_0} \right) \right) + s^4 \left( \frac{22}{3} \nu \log \left( \frac{z_i}{z'_0} \right) \right) \right]. \quad (5.8)$$

#### 2. Case II: Infall from infinity

By inserting  $s = z/z_i$  back in the Eq. (5.8) and taking the limit when  $z_i \rightarrow \infty$ , the expression for “ $\delta_{(\text{SH} \rightarrow \text{MH})} d\mathcal{E}/dt$ ” takes the form

$$\delta_{(\text{SH} \rightarrow \text{MH})} (d\mathcal{E}/dt)_{\text{inst}} = \frac{16 c^5}{15 G} \nu^2 \gamma^8 \left[ -\frac{110}{3} \nu \log \left( \frac{z}{z'_0} \right) \right]. \quad (5.9)$$

### C. The 3PN instantaneous Energy Flux in ADM Coordinates

In this section we provide the expressions for instantaneous part of the energy flux in ADM coordinates which could be useful for the comparison with the numerical relativity results. One can write the energy flux  $\mathcal{F}$  in the ADM coordinates as

$$\mathcal{F}_{\text{ADM}} = \mathcal{F}_{\text{SH}} + \delta_{(\text{SH} \rightarrow \text{ADM})} \mathcal{F}, \quad (5.10)$$

where  $\mathcal{F}_{\text{SH}}$  is given by Eqs. (5.1) and (5.2) and the shift “ $\delta_{(\text{SH} \rightarrow \text{ADM})} \mathcal{F}$ ” for the head-on situation is

$$\delta_{(\text{SH} \rightarrow \text{ADM})} \mathcal{F} = -\frac{G^4 m^5 \nu^2}{c^9 z^5} \left\{ -\frac{56}{15} \dot{z}^4 \nu + \frac{G m}{z} \dot{z}^2 \left( \frac{4}{5} + \frac{88}{15} \nu \right) \right\}$$

$$\begin{aligned}
& -\frac{G^4 m^5 \nu^2}{c^{11} z^5} \left\{ \dot{z}^6 \left( \frac{628}{45} \nu - \frac{116}{45} \nu^2 \right) \right. \\
& + \frac{Gm}{z} \dot{z}^4 \left( -\frac{512}{105} - \frac{24946}{315} \nu + \frac{386}{315} \nu^2 \right) \\
& + \frac{G^2 m^2}{z^2} \dot{z}^2 \left( \frac{36}{5} + \left[ \frac{4568}{1575} - \frac{14}{5} \pi^2 + \frac{1408}{45} \log \left( \frac{z}{z_0'} \right) \right] \nu + \frac{132}{5} \nu^2 \right) \\
& \left. + \frac{G^3 m^3}{z^3} \left( \frac{16}{35} + \frac{128}{35} \nu - \frac{768}{35} \nu^2 \right) \right\}. \tag{5.11}
\end{aligned}$$

We have made use of Eq. (3.1) to get Eq. (5.11) from the general expression for “ $\delta_{(\text{SH} \rightarrow \text{ADM})} \mathcal{F}$ ”, given by Eq. (6.11) of [25].

### 1. Case I: Infall from a finite distance

Substituting for  $\dot{z}$  in ADM coordinates in the expression for energy flux in ADM coordinates, we obtain the expression for the difference “ $\delta_{(\text{SH} \rightarrow \text{ADM})} d\mathcal{E}/dt$ ,” which should be added to Eq. (5.4) to obtain the instantaneous part of the energy flux in ADM coordinates. It reads

$$\begin{aligned}
\delta_{(\text{SH} \rightarrow \text{ADM})} (d\mathcal{E}/dt)_{\text{inst}} &= \frac{16 c^5}{15 G} \nu^2 \gamma^5 \left\{ \left[ -\frac{5}{4} + \frac{5}{2} \nu + s \left( 1 - \frac{39}{2} \nu \right) + 14 s^2 \nu + s^3 \left( \frac{1}{4} + 3 \nu \right) \right] \gamma^2 \right. \\
& + \sqrt{2} \sqrt{1-s} \left[ \frac{56}{15} \nu - \frac{16}{5} s \nu \right] \gamma^{5/2} \\
& + \left[ \frac{129}{14} + \left[ \frac{7355}{168} + \frac{105}{32} \pi^2 - \frac{110}{3} \log \left( \frac{z}{z_0'} \right) \right] \nu + \frac{482}{21} \nu^2 \right. \\
& + s \left( -\frac{145}{7} + \left[ \frac{20641}{210} - \frac{21}{8} \pi^2 + \frac{88}{3} \log \left( \frac{z}{z_0'} \right) \right] \nu - \frac{18875}{84} \nu^2 \right) \\
& + s^2 \left( \frac{71}{7} - \frac{11729}{42} \nu + \frac{31027}{84} \nu^2 \right) + s^3 \left( -\frac{29}{7} + \frac{7361}{84} \nu - \frac{1873}{21} \nu^2 \right) \\
& \left. \left. + s^4 \left( \frac{71}{14} + \left[ \frac{12917}{280} - \frac{21}{32} \pi^2 + \frac{22}{3} \log \left( \frac{z_i}{z_0'} \right) \right] \nu - \frac{405}{7} \nu^2 \right) \right] \gamma^3 \right\}. \tag{5.12}
\end{aligned}$$

### 2. Case II: Infall from infinity

Inserting  $s = z/z_i$  back in Eq. (5.12) and taking the limit when  $z_i \rightarrow \infty$  we obtain the expression for “ $\delta_{(\text{SH} \rightarrow \text{ADM})} (d\mathcal{E}/dt)$ ” as

$$\begin{aligned}
\delta_{(\text{SH} \rightarrow \text{ADM})} (d\mathcal{E}/dt)_{\text{inst}} &= \frac{16 c^5}{15 G} \nu^2 \gamma^5 \left\{ \left[ -\frac{5}{4} + \frac{5}{2} \nu \right] \gamma^2 + \frac{56 \sqrt{2}}{15} \nu \gamma^{5/2} \right. \\
& \left. + \left[ \frac{129}{14} + \left[ \frac{7355}{168} + \frac{105}{32} \pi^2 - \frac{110}{3} \log \left( \frac{z}{z_0'} \right) \right] \nu + \frac{482}{21} \nu^2 \right] \gamma^3 \right\}. \tag{5.13}
\end{aligned}$$

## VI. HEREDITARY CONTRIBUTIONS IN THE FLUX FOR HEAD-ON COLLISION

In this section we shall compute the hereditary contributions to the GW energy flux at 3PN order given by the Eqs. (2.9), Eqs. (2.14), and (2.15). The first hereditary contribution to the energy flux occurs at 1.5PN order and is due to GW tails caused by the interaction between mass quadrupole moment and the ADM mass of the source causing the spacetime curvature. This contribution is given by the first term in Eq. (2.14) where as the second term represents the subdominant tail at 2.5PN order caused due to interaction of a higher order multipole moment with the ADM mass of the source. Two cubic order tail terms, given by Eq. (2.15), known as tails-of-tails and tail-squared



occur at 3PN order and are caused due to the interaction of tails with ADM mass of the source and interaction of tails among themselves.

It should be evident from Eqs. (2.14) and (2.15) that the computation of all terms would require only Newtonian order inputs except the mass-type quadrupolar tail term— first term in Eq. (2.14) —which would include 1PN corrections. Note that the second term appearing in Eq. (2.14) and needed to be evaluated with Newtonian accuracy, does not contribute to the energy flux for the case of radial infall from infinity. The reason is that this term involves 4th and 6th derivatives of octupole moment [see Eq. (4.3) for the expression] but one can check that, at the Newtonian order third and higher order derivatives of octupole moment vanish (after we substitute for the Newtonian  $\dot{z}$  appearing in corresponding expressions), i.e.  $I_{ijk}^{(n)} = 0$  for  $n > 2$ , and hence the second integral would not contribute. In this case we only have the first term left in Eq. (2.14) which gives a hereditary contribution to GW energy flux at 2.5PN order. Computation of this term will require the knowledge of 1PN accurate expression for the quadrupole mass moment and 1PN accurate trajectory of the system. In addition to this the 1PN accurate expression for ADM mass would also be needed for the computations of tails at 2.5PN order. We have provided the 1PN trajectory in Sec. IIID which will be used in computing the hereditary contributions. The second term of Eq. (2.14) survives for the case of infall from a finite initial separation and thus must be taken into account while computing the tails at 2.5PN order for the finite initial separation case. As for the instantaneous part we will compute the hereditary contributions as well for two different situations, case I- infall from a finite distance and case II-infall from infinity.

It is important to note that, at 3PN order unlike the instantaneous part of the flux, the hereditary part is the same in all the three coordinate systems – SH, MH and ADM since it involves the inputs which are at most required at 1PN order and are same in all three coordinate systems.

In addition to the inputs listed in Sec IIID we also need 1PN accurate expressions for mass quadrupole moment and for ADM mass. The mass quadrupole moment in terms of the variables  $z$  and  $\dot{z}$  is given by Eq. (4.2). In the 1PN limit it reads,

$$I_{ij} = \nu m z^2 \left\{ 1 + \frac{1}{c^2} \left[ \dot{z}^2 \left( \frac{9}{14} - \frac{27}{14} \nu \right) + \frac{Gm}{z} \left( -\frac{5}{7} + \frac{8}{7} \nu \right) \right] \right\} n_{\langle i} n_{j \rangle}, \quad (6.1)$$

The relation between the ADM mass  $M$  and total mass  $m = m_1 + m_2$  is given by,

$$M = m \left[ 1 + \frac{\nu}{c^2} \left( \frac{\dot{z}^2}{2} - \frac{Gm}{z} \right) \right], \quad (6.2)$$

where  $\dot{z}$  is given by Eq. (3.8) and is needed to be just Newtonian accurate.

### A. Case I: Infall from a finite distance

In this case the 1PN accurate expression for quadrupole moment and Newtonian accurate expression for octupole moment take the form,

$$I_{ij} = m z^2 \nu \left\{ 1 + \frac{Gm}{c^2 z} \left[ \frac{4}{7} - \frac{19}{7} \nu + s \left( -\frac{9}{7} + \frac{27}{7} \nu \right) \right] \right\} n_{\langle i} n_{j \rangle}, \quad (6.3a)$$

$$I_{ijk} = -\nu m z^3 \sqrt{1 - 4\nu} n_{\langle i} n_j n_{k \rangle}. \quad (6.3b)$$

The relation between ADM mass and total mass at 1PN order is given as,

$$M = m \left[ 1 - \frac{Gm\nu}{c^2 z_i} \right]. \quad (6.4)$$

From the above expression the ADM mass  $M$  is independent of  $z$  which is consistent with the constancy of  $M$  and the recognition of the expression of the initial Newtonian energy  $-Gm\nu/z_i$ .

Hereditary terms will involve two integrals, which are given as

$$I_{\text{tail}}^{\text{Quadrupole}} = \int_{u(z_i)}^u d\tau I_{ij}^{(5)}(\tau) \left[ \log \left( \frac{c}{2z} (u - \tau) \right) + \frac{11}{12} \right], \quad (6.5a)$$

$$I_{\text{tail}}^{\text{Octupole}} = \int_{u(z_i)}^u d\tau I_{ijk}^{(6)}(\tau) \left[ \log \left( \frac{c}{2z} (u - \tau) \right) + \frac{97}{60} \right], \quad (6.5b)$$

$$I_{\text{tail}(\text{tail})} = \int_{u(z_i)}^u d\tau I_{ij}^{(6)}(\tau) \left[ \log^2 \left( \frac{c}{2z} (u - \tau) \right) + \frac{57}{70} \log \left( \frac{c}{2z} (u - \tau) \right) + \frac{124627}{44100} \right]. \quad (6.5c)$$

After evaluating the integrals we get

$$\begin{aligned} I_{\text{tail}}^{\text{Quadrupole}} = & \frac{G^2 m^3 \nu}{z^4} \left\{ \left[ \frac{55}{6} - 5 \log(8\gamma) + s \left( -\frac{22}{3} + 4 \log(8\gamma) \right) + s^4 \left( -\frac{11}{6} + \log(8\gamma) + 2 \text{Int1}(s) \right) \right] \right. \\ & + \left[ \frac{17}{3} (1 - \nu) \left( -11 + 6 \log(8\gamma) \right) + s \left[ \left( -\frac{19}{2} + \frac{67}{6} \nu \right) \left( -11 + 6 \log(8\gamma) \right) \right] \right. \\ & + s^2 \left[ \left( \frac{80}{21} - \frac{107}{21} \nu \right) \left( -11 + 6 \log(8\gamma) \right) \right] + s^5 \left[ \left( \frac{1}{42} - \frac{17}{42} \nu \right) \left( -11 + 6 \log(8\gamma) \right) \right. \\ & \left. \left. - 2 \text{Int20}(s) + \text{Int30}(s) - \text{Int40}(s) + \nu (-2 \text{Int21}(s) + \text{Int31}(s) - \text{Int41}(s)) \right] \right] \gamma \Big\} n_{\langle i} n_{j \rangle}, \quad (6.6a) \end{aligned}$$

$$I_{\text{tail}}^{\text{Octupole}} = \frac{G^{5/2} m^{7/2} \nu}{z^{9/2}} \sqrt{1 - 4\nu} \left\{ \sqrt{2} \sqrt{1 - s} \left[ s \left( -\frac{194}{5} + 12 \log(8\gamma) \right) - \frac{12s^{9/2}}{\sqrt{1 - s}} \text{Int7}(s) \right] \right\} n_{\langle i} n_j n_{k \rangle}, \quad (6.6b)$$

$$\begin{aligned} I_{\text{tail}(\text{tail})} = & \frac{G^{5/2} m^{7/2} \nu}{z^{11/2}} \sqrt{2} \sqrt{1 - s} \left\{ \frac{249254}{2205} - \frac{114}{7} \log(8\gamma) + 10 \log^2(8\gamma) + s \left( -\frac{249254}{3675} + \frac{342}{35} \log(8\gamma) \right. \right. \\ & \left. \left. - 6 \log^2(8\gamma) \right) + \frac{s^{11/2}}{\sqrt{1 - s}} \left( 2 \text{Int5}(s) + \frac{57}{35} \text{Int6}(s) - 2 \text{Int6}(s) \log(8\gamma) \right) \right\} n_{\langle i} n_{j \rangle}. \quad (6.6c) \end{aligned}$$

where  $\text{Int1}(s)$ ,  $\text{Int20}(s)$ ,  $\text{Int21}(s)$ ,  $\text{Int30}(s)$ ,  $\text{Int31}(s)$ ,  $\text{Int40}(s)$ , and  $\text{Int41}(s)$  appearing in Eq. (6.6a) and  $\text{Int7}(s)$  appearing in Eq. (6.6b) are given by

$$\text{Int1}(s) = 4 \int_s^1 \left( \frac{5 - 3y}{y^5} \right) \log \left[ s^{-3/2} (g(s) - g(y)) \right] dy, \quad (6.7a)$$

$$\text{Int20}(s) = \int_s^1 \left( \frac{1540 - 1876y + 522y^2}{7y^6} \right) \log \left[ s^{-3/2} (g(s) - g(y)) \right] dy, \quad (6.7b)$$

$$\text{Int21}(s) = \int_s^1 \left( \frac{-1365 + 2296y - 831y^2}{7y^6} \right) \log \left[ s^{-3/2} (g(s) - g(y)) \right] dy, \quad (6.7c)$$

$$\text{Int30}(s) = 4 \int_s^1 \left( \frac{(5 - 3y)(5 - y)}{y^6} \right) \log \left[ s^{-3/2} (g(s) - g(y)) \right] dy, \quad (6.7d)$$

$$\text{Int31}(s) = 2 \int_s^1 \left( \frac{(5 - 3y)(-5 + 9y)}{y^6} \right) \log \left[ s^{-3/2} (g(s) - g(y)) \right] dy, \quad (6.7e)$$

$$\text{Int40}(s) = 4 \int_s^1 \left( \frac{5 - 3y}{y^5} \right) \left( \frac{h_0(s) - h_0(y)}{g(s) - g(y)} \right) dy, \quad (6.7f)$$

$$\text{Int41}(s) = 4 \int_s^1 \left( \frac{5 - 3y}{y^5} \right) \left( \frac{h_1(s) - h_1(y)}{g(s) - g(y)} \right) dy. \quad (6.7g)$$

$$\text{Int7}(s) = \int_s^1 \left( \frac{7-6y}{y^{9/2}\sqrt{1-y}} \right) \log \left[ s^{-3/2} (g(s) - g(y)) \right] dy, \quad (6.7h)$$

and  $\text{Int5}(s)$  and  $\text{Int6}(s)$  appearing in Eq. (6.6c) are given by

$$\text{Int5}(s) = \int_s^1 \frac{1}{\sqrt{1-y}} \left( \frac{110-154y+48y^2}{y^{13/2}} \right) \log^2 \left[ s^{-3/2} (g(s) - g(y)) \right] dy, \quad (6.8a)$$

$$\text{Int6}(s) = \int_s^1 \frac{1}{\sqrt{1-y}} \left( \frac{110-154y+48y^2}{y^{13/2}} \right) \log \left[ s^{-3/2} (g(s) - g(y)) \right] dy. \quad (6.8b)$$

With Eq. (6.6) we can write the various pieces of the hereditary contributions to GW energy flux given by Eq. (2.9) as

$$\begin{aligned} \mathcal{F}_{\text{tail}} = & \frac{16}{15} \frac{c^5}{G} \nu^2 \gamma^5 \left\{ \sqrt{2}\sqrt{1-s} \left[ \frac{55}{6} - 5 \log(8\gamma) + s \left( -\frac{22}{3} + 4 \log(8\gamma) \right) + s^4 \left( -\frac{11}{6} + 2 \text{Int1}(s) + \log(8\gamma) \right) \right] \gamma^{3/2} \right. \\ & + \sqrt{2}\sqrt{1-s} \left[ -\frac{7601}{84} + \frac{16577}{168} \nu + \left( \frac{691}{14} - \frac{1507}{28} \nu \right) \log(8\gamma) + s \left( \frac{4895}{28} - \frac{35365}{168} \nu \right. \right. \\ & + \left. \left. \left( -\frac{1335}{14} + \frac{3215}{28} \nu \right) \log(8\gamma) \right) + s^2 \left( -\frac{8027}{105} + \frac{18511}{210} \nu + \left( \frac{298}{7} - \frac{361}{7} \nu \right) \log(8\gamma) \right) + s^4 \left( \frac{473}{84} - \frac{407}{56} \nu \right. \right. \\ & + \left. \left. \left( -\frac{43}{7} + \frac{111}{14} \nu \right) \text{Int1}(s) + \left( -\frac{43}{14} + \frac{111}{28} \nu \right) \log(8\gamma) \right) + s^5 \left( -\frac{275}{28} + \frac{2717}{168} \nu + \left( \frac{73}{7} - \frac{179}{14} \nu \right) \text{Int1}(s) \right. \right. \\ & - 2 \text{Int20}(s) + \text{Int30}(s) - \text{Int40}(s) + \nu (-2 \text{Int21}(s) + \text{Int31}(s) - \text{Int41}(s)) + \left( \frac{75}{14} - \frac{247}{28} \nu \right) \log(8\gamma) \Big) \\ & \left. + \frac{s^{11/2}}{\sqrt{1-s}} \left( \frac{8}{7} - \frac{32}{7} \nu \right) \text{Int7}(s) \right] \gamma^{5/2} \Big\}, \quad (6.9) \end{aligned}$$

$$\begin{aligned} \mathcal{F}_{\text{tail(tail)}} = & \frac{16}{15} \frac{c^5}{G} \nu^2 \gamma^5 \left\{ \left[ \left( \frac{498508}{2205} - \frac{228}{7} \log(8\gamma) + 20 \log^2(8\gamma) - \frac{1712}{21} \log\left(\frac{z}{z_0}\right) \right) + s \left( -\frac{3988064}{11025} \right. \right. \right. \\ & + \frac{1824}{35} \log(8\gamma) - 32 \log^2(8\gamma) + \frac{13696}{105} \log\left(\frac{z}{z_0}\right) \Big) + s^2 \left( \frac{498508}{3675} - \frac{684}{35} \log(8\gamma) + 12 \log^2(8\gamma) \right. \\ & \left. \left. - \frac{1712}{35} \log\left(\frac{z}{z_0}\right) \right) + s^{11/2} \sqrt{1-s} \left( 4 \text{Int5}(s) + \left( \frac{114}{35} - 4 \log(8\gamma) \right) \text{Int6}(s) \right) \right] \gamma^3 \Big\}, \quad (6.10) \end{aligned}$$

$$\begin{aligned} \mathcal{F}_{(\text{tail})^2} = & \frac{16}{15} \frac{c^5}{G} \nu^2 \gamma^5 \left\{ \left[ \frac{3025}{72} - \frac{275}{6} \log(8\gamma) + \frac{25}{2} \log^2(8\gamma) + s \left( -\frac{605}{9} + \frac{220}{3} \log(8\gamma) - 20 \log^2(8\gamma) \right) \right. \right. \\ & + s^2 \left( \frac{242}{9} - \frac{88}{3} \log(8\gamma) + 8 \log^2(8\gamma) \right) + s^4 \left( -\frac{605}{36} + \frac{55}{3} \text{Int1}(s) + \left( \frac{55}{3} - 10 \text{Int1}(s) \right) \log(8\gamma) \right. \\ & - 5 \log^2(8\gamma) \Big) + s^5 \left( \frac{121}{9} - \frac{44}{3} \text{Int1}(s) + \left( -\frac{44}{3} + 8 \text{Int1}(s) \right) \log(8\gamma) + 4 \log^2(8\gamma) \right) \\ & \left. \left. + s^8 \left( \frac{121}{72} - \frac{11}{3} \text{Int1}(s) + 2 [\text{Int1}(s)]^2 + \left( -\frac{11}{6} + 2 \text{Int1}(s) \right) \log(8\gamma) + \frac{1}{2} \log^2(8\gamma) \right) \right] \gamma^3 \right\}. \quad (6.11) \end{aligned}$$

Combining Eqs. (2.9), (6.9), (6.10) and (6.11) now we can write the hereditary contribution at 3PN order involving contribution from tails, tail-of-tail and tail squared terms as

$$\begin{aligned} \mathcal{F}_{\text{hered}} = & \frac{16}{15} \frac{c^5}{G} \nu^2 \gamma^5 \left\{ \sqrt{2}\sqrt{1-s} \left[ \frac{55}{6} - 5 \log(8\gamma) + s \left( -\frac{22}{3} + 4 \log(8\gamma) \right) + s^4 \left( -\frac{11}{6} + 2 \text{Int1}(s) + \log(8\gamma) \right) \right] \gamma^{3/2} \right. \\ & + \sqrt{2}\sqrt{1-s} \left[ -\frac{7601}{84} + \frac{16577}{168} \nu + \left( \frac{691}{14} - \frac{1507}{28} \nu \right) \log(8\gamma) + s \left( \frac{4895}{28} - \frac{35365}{168} \nu \right. \right. \end{aligned}$$

$$\begin{aligned}
& + \left( -\frac{1335}{14} + \frac{3215}{28} \nu \right) \log(8\gamma) + s^2 \left( -\frac{8027}{105} + \frac{18511}{210} \nu + \left( \frac{298}{7} - \frac{361}{7} \nu \right) \log(8\gamma) \right) + s^4 \left( \frac{473}{84} - \frac{407}{56} \nu \right. \\
& + \left( -\frac{43}{7} + \frac{111}{14} \nu \right) \text{Int1}(s) + \left( -\frac{43}{14} + \frac{111}{28} \nu \right) \log(8\gamma) + s^5 \left( -\frac{275}{28} + \frac{2717}{168} \nu + \left( \frac{73}{7} - \frac{179}{14} \nu \right) \text{Int1}(s) \right. \\
& - 2 \text{Int20}(s) + \text{Int30}(s) - \text{Int40}(s) + \nu (-2 \text{Int21}(s) + \text{Int31}(s) - \text{Int41}(s)) + \left. \left( \frac{75}{14} - \frac{247}{28} \nu \right) \log(8\gamma) \right) \\
& + \frac{s^{11/2}}{\sqrt{1-s}} \left( \frac{8}{7} - \frac{32}{7} \nu \right) \text{Int7}(s) \Big] \gamma^{5/2} \\
& + \left[ \frac{4729189}{17640} - \frac{3293}{42} \log(8\gamma) + \frac{65}{2} \log^2(8\gamma) - \frac{1712}{21} \log\left(\frac{z}{z_0}\right) + s \left( -\frac{4729189}{11025} + \frac{13172}{105} \log(8\gamma) \right. \right. \\
& - 52 \log^2(8\gamma) + \frac{13696}{105} \log\left(\frac{z}{z_0}\right) + s^2 \left( \frac{1791974}{11025} - \frac{5132}{105} \log(8\gamma) + 20 \log^2(8\gamma) - \frac{1712}{35} \log\left(\frac{z}{z_0}\right) \right) \\
& + s^4 \left( -\frac{605}{36} + \frac{55}{3} \text{Int1}(s) + \left( \frac{55}{3} - 10 \text{Int1}(s) \right) \log(8\gamma) - 5 \log^2(8\gamma) \right) + s^5 \left( \frac{121}{9} - \frac{44}{3} \text{Int1}(s) \right. \\
& + \left( -\frac{44}{3} + 8 \text{Int1}(s) \right) \log(8\gamma) + 4 \log^2(8\gamma) + s^{11/2} \sqrt{1-s} \left( 4 \text{Int5}(s) + \left( \frac{114}{35} - 4 \log(8\gamma) \right) \text{Int6}(s) \right) \\
& \left. \left. + s^8 \left( \frac{121}{72} - \frac{11}{3} \text{Int1}(s) + 2 [\text{Int1}(s)]^2 + \left( -\frac{11}{6} + 2 \text{Int1}(s) \right) \log(8\gamma) + \frac{1}{2} \log^2(8\gamma) \right) \right] \gamma^3 \right\}. \quad (6.12)
\end{aligned}$$

As we can see the above equation still has some dependence on the arbitrary scale  $z_0$  at 3PN order. Recall, the presence of a logarithmic dependence on  $z_0$  in the instantaneous contribution to the energy flux at 3PN order. The term appearing in the hereditary contribution exactly cancels with similar terms present in instantaneous flux expression for energy flux and thus the total flux becomes independent of the arbitrary length scale  $z_0$  as expected.

## B. Case II: Infall from infinity

The 1PN accurate expression for quadrupole moment reads

$$I_{ij} = m z^2 \nu \left[ 1 + \frac{Gm}{c^2 z} \left( \frac{4}{7} - \frac{19}{7} \nu \right) \right] n_{\langle i} n_{j \rangle}. \quad (6.13)$$

It is evident from Eq. (3.25), the relation connecting ADM mass  $M$  and total mass  $m$  [Eq. (6.2)] reduces to

$$M = m. \quad (6.14)$$

This is consistent with the earlier comment and corresponds to energy vanishing initially. In order to compute the hereditary contribution first we need to evaluate the two integrals appearing in Eqs. (2.14) and (2.15). The integral associated with the first term of Eq. (2.14) is

$$I_{\text{tail}} = \int_0^{+\infty} d\tau I_{ij}^{(5)}(u - \tau) \left[ \log\left(\frac{c\tau}{2z}\right) + \frac{11}{12} \right], \quad (6.15)$$

and the integral appearing in the first term of Eq. (2.15) is

$$I_{\text{tail(tail)}} = \int_0^{+\infty} d\tau I_{ij}^{(6)}(u - \tau) \left[ \log^2\left(\frac{c\tau}{2z}\right) + \frac{57}{70} \log\left(\frac{c\tau}{2z}\right) + \frac{124627}{44100} \right]. \quad (6.16)$$

Having all the relevant inputs at the required PN order the value of these integrals read,

$$\begin{aligned}
I_{\text{tail}} = & \frac{G^2 m^3 \nu}{z^4} \left\{ \left[ -\frac{71}{6} - \frac{5}{\sqrt{3}} \pi - 5 \log\left(\frac{2}{3} \gamma\right) \right] \right. \\
& \left. + \left[ -\frac{2497}{21} + \frac{166}{\sqrt{3}} \pi + \left[ -\frac{2161}{42} - 22\sqrt{3} \pi \right] \nu + 34(1 - \nu) \log\left(\frac{2}{3} \gamma\right) \right] \gamma \right\} n_{\langle i} n_{j \rangle}, \quad (6.17a)
\end{aligned}$$

$$I_{\text{tail(tail)}} = \frac{G^{5/2} m^{7/2} \nu}{\sqrt{2} z^{11/2}} \left\{ \frac{4894237}{8820} + \frac{386\sqrt{3}}{7} \pi + 20 \pi^2 + \left( \frac{1158}{7} + \frac{40}{\sqrt{3}} \pi \right) \log \left( \frac{2}{3} \gamma \right) \right. \\ \left. + 20 \log^2 \left( \frac{2}{3} \gamma \right) + 80 \psi^{(1)} \left( \frac{11}{3} \right) \right\} n_{\langle i} n_{j \rangle}, \quad (6.17b)$$

where the quantity  $\psi^{(1)} \left( \frac{11}{3} \right)$  appearing in Eq. (6.17b) is a PolyGamma function whose numerical value is 0.313 25. (Of course, formally they correspond to  $s \rightarrow 0$  case of the previous section.)

Using Eq. (6.17) in Eqs. (2.14)-(2.15) we write various pieces of hereditary contribution given by Eq. (2.9) as

$$\mathcal{F}_{\text{tail}} = \frac{16 c^5}{15 G} \nu^2 \gamma^5 \left\{ \sqrt{2} \left[ -\frac{71}{6} - \frac{5}{\sqrt{3}} \pi - 5 \log \left( \frac{2}{3} \gamma \right) \right] \gamma^{3/2} \right. \\ \left. + \sqrt{2} \left[ -\frac{6935}{84} + \frac{2539}{14\sqrt{3}} \pi + \left( -\frac{16525}{168} - \frac{801\sqrt{3}}{28} \pi \right) \nu + \left( \frac{691}{14} - \frac{1507}{28} \nu \right) \log \left( \frac{2}{3} \gamma \right) \right] \gamma^{5/2} \right\}. \quad (6.18)$$

As mentioned earlier the 2PN accurate energy flux has been given in [16] which involves the hereditary contribution to the energy flux at 1.5PN order. On comparing our results [1.5PN term in Eq. (6.18) above with coefficient -5] and [Eq. (2.31) of [16] with coefficient -15] for the contribution due to dominant tail we find a mismatch. This apparent discrepancy is a gauge-artifact arising from the difference in our choice of  $u = t - r/c$  in contrast to the choice in SPW [16]  $u_{\text{SPW}} = t - r/c - (2GM/c^3) \log(c^2 r/Gm)$ . We have checked that once we adopt the SPW definition of retarded time in harmonic coordinates  $u_{\text{SPW}}$ , our result also leads to the coefficient  $-15$  as in [16]. This difference serves to remind us that the representation of the energy flux in terms of  $\gamma$  is *not* gaugeinvariant.

$$\mathcal{F}_{\text{tail(tail)}} = \frac{16 c^5}{15 G} \nu^2 \gamma^5 \left\{ \left[ \frac{4894237}{8820} + \frac{386\sqrt{3}}{7} \pi + 20 \pi^2 - \frac{1712}{21} \log \left( \frac{z}{z_0} \right) + \left( \frac{1158}{7} + \frac{40}{\sqrt{3}} \pi \right) \log \left( \frac{2}{3} \gamma \right) \right. \right. \\ \left. \left. + 20 \log^2 \left( \frac{2}{3} \gamma \right) + 80 \psi^{(1)} \left( \frac{11}{3} \right) \right] \gamma^3 \right\}, \quad (6.19)$$

$$\mathcal{F}_{(\text{tail})^2} = \frac{16 c^5}{15 G} \nu^2 \gamma^5 \left\{ \left[ \frac{5041}{72} + \frac{355}{6\sqrt{3}} \pi + \frac{25}{6} \pi^2 + \left( \frac{355}{6} + \frac{25}{\sqrt{3}} \pi \right) \log \left( \frac{2}{3} \gamma \right) + \frac{25}{2} \log^2 \left( \frac{2}{3} \gamma \right) \right] \gamma^3 \right\}. \quad (6.20)$$

Now we can write the total hereditary contribution up to 3PN order to the energy flux as

$$\mathcal{F}_{\text{hered}} = \frac{16 c^5}{15 G} \nu^2 \gamma^5 \left\{ \sqrt{2} \left[ -\frac{71}{6} - \frac{5}{\sqrt{3}} \pi - 5 \log \left( \frac{2}{3} \gamma \right) \right] \gamma^{3/2} \right. \\ + \sqrt{2} \left[ -\frac{6935}{84} + \frac{2539}{14\sqrt{3}} \pi + \left( -\frac{16525}{168} - \frac{801\sqrt{3}}{28} \pi \right) \nu + \left( \frac{691}{14} - \frac{1507}{28} \nu \right) \log \left( \frac{2}{3} \gamma \right) \right] \gamma^{5/2} \\ + \left[ \frac{11023519}{17640} + \frac{9433}{42\sqrt{3}} \pi + \frac{145}{6} \pi^2 - \frac{1712}{21} \log \left( \frac{z}{z_0} \right) + \left( \frac{9433}{42} + \frac{65}{\sqrt{3}} \pi \right) \log \left( \frac{2}{3} \gamma \right) \right. \\ \left. \left. + \frac{65}{2} \log^2 \left( \frac{2}{3} \gamma \right) + 80 \psi^{(1)} \left( \frac{11}{3} \right) \right] \gamma^3 \right\}. \quad (6.21)$$

The presence of the arbitrary scale  $z_0$  in the above expression is similar to the one already noted in Eq. (6.12) and will disappear from the final expression for energy flux.

## VII. THE COMPLETE 3PN ENERGY FLUX FOR HEAD-ON SITUATION

### A. Case I: Infall from finite a distance

Having computed both the instantaneous and the hereditary contributions to the energy flux at 3PN order for head-on situation we are now ready to write the complete 3PN far-zone energy flux due to head-on infall of two compact

objects with arbitrary mass ratios. Since the ADM coordinates are independent of gauge-dependent logarithms they may be better suited for comparison with numerical relativity results, and we exhibit the complete 3PN accurate energy flux expression in these coordinates obtained by adding the hereditary part [Eq. (6.12)] and instantaneous part [Eq. (5.12)] of the energy flux. The final result is

$$\begin{aligned}
\left(\frac{d\mathcal{E}}{dt}\right)_{\text{ADM}} = & \frac{16}{15} \frac{c^5}{G} \nu^2 \gamma^5 \left\{ 1 - s + \left[ -\frac{43}{7} + \frac{111}{14} \nu + s \left( \frac{116}{7} - \frac{131}{7} \nu \right) + s^2 \left( -\frac{71}{7} + \frac{135}{14} \nu \right) \right] \gamma \right. \\
& + \sqrt{2} \sqrt{1-s} \left[ \frac{55}{6} - 5 \log(8\gamma) + s \left( -\frac{22}{3} + 4 \log(8\gamma) \right) + s^4 \left( -\frac{11}{6} + 2 \text{Int1}(s) + \log(8\gamma) \right) \right] \gamma^{3/2} \\
& + \left[ -\frac{4643}{108} - \frac{713}{36} \nu + \frac{112}{3} \nu^2 + s \left( -\frac{4282}{189} + \frac{28505}{126} \nu - \frac{2864}{21} \nu^2 \right) + s^2 \left( \frac{1870}{21} - \frac{5251}{18} \nu + \frac{8800}{63} \nu^2 \right) \right. \\
& + s^3 \left( -\frac{329}{12} + \frac{1219}{12} \nu - \frac{872}{21} \nu^2 \right) \left. \right] \gamma^2 + \sqrt{2} \sqrt{1-s} \left[ -\frac{7601}{84} + \frac{11651}{120} \nu + \left( \frac{691}{14} - \frac{1507}{28} \nu \right) \log(8\gamma) \right. \\
& + s \left( \frac{4895}{28} - \frac{173113}{840} \nu + \left( -\frac{1335}{14} + \frac{3215}{28} \nu \right) \log(8\gamma) \right) \\
& + s^2 \left( -\frac{8027}{105} + \frac{18031}{210} \nu + \left( \frac{298}{7} - \frac{361}{7} \nu \right) \log(8\gamma) \right) \\
& + s^4 \left( \frac{473}{84} - \frac{407}{56} \nu + \left( -\frac{43}{7} + \frac{111}{14} \nu \right) \text{Int1}(s) + \left( -\frac{43}{14} + \frac{111}{28} \nu \right) \log(8\gamma) \right) \\
& + s^5 \left( -\frac{275}{28} + \frac{2717}{168} \nu + \left( \frac{73}{7} - \frac{179}{14} \nu \right) \text{Int1}(s) - 2 \text{Int20}(s) + \text{Int30}(s) - \text{Int40}(s) \right. \\
& + \nu (-2 \text{Int21}(s) + \text{Int31}(s) - \text{Int41}(s)) + \left( \frac{75}{14} - \frac{247}{28} \nu \right) \log(8\gamma) + \frac{s^{11/2}}{\sqrt{1-s}} \left( \frac{8}{7} - \frac{32}{7} \nu \right) \text{Int7}(s) \left. \right] \gamma^{5/2} \\
& + \left[ \frac{155961373}{582120} + \left[ -\frac{5467459}{5544} - \frac{1125}{32} \pi^2 \right] \nu + \frac{1865377}{5544} \nu^2 + \frac{231520}{2079} \nu^3 - \frac{3293}{42} \log(8\gamma) + \frac{65}{2} \log^2(8\gamma) \right. \\
& + s \left( -\frac{8236351}{40425} + \left[ \frac{5608279}{8316} + \frac{819}{16} \pi^2 \right] \nu + \frac{1302751}{1386} \nu^2 - \frac{1366537}{2079} \nu^3 + \frac{13172}{105} \log(8\gamma) - 52 \log^2(8\gamma) \right) \\
& + s^2 \left( -\frac{74317681}{363825} + \left[ \frac{1312231}{924} - \frac{123}{8} \pi^2 \right] \nu - \frac{252311}{84} \nu^2 + \frac{304961}{297} \nu^3 - \frac{5132}{105} \log(8\gamma) + 20 \log^2(8\gamma) \right) \\
& + s^3 \left( \frac{207379}{1155} - \frac{4141933}{2772} \nu + \frac{6033829}{2772} \nu^2 - \frac{406498}{693} \nu^3 \right) + s^4 \left( -\frac{77347}{2772} + \left[ \frac{74357}{264} - \frac{21}{32} \pi^2 \right] \nu - \frac{848843}{1848} \nu^2 \right. \\
& + \frac{8076}{77} \nu^3 + \frac{55}{3} \text{Int1}(s) + \left( \frac{55}{3} - 10 \text{Int1}(s) \right) \log(8\gamma) - 5 \log^2(8\gamma) \left. \right) + s^5 \left( \frac{121}{9} - \frac{44}{3} \text{Int1}(s) \right. \\
& + \left( -\frac{44}{3} + 8 \text{Int1}(s) \right) \log(8\gamma) + 4 \log^2(8\gamma) \left. \right) + s^{11/2} \sqrt{1-s} \left( 4 \text{Int5}(s) + \left( \frac{114}{35} - 4 \log(8\gamma) \right) \text{Int6}(s) \right) \\
& + s^8 \left( \frac{121}{72} - \frac{11}{3} \text{Int1}(s) + 2 [\text{Int1}(s)]^2 + \left( -\frac{11}{6} + 2 \text{Int1}(s) \right) \log(8\gamma) + \frac{1}{2} \log^2(8\gamma) \right) \left. \right] \gamma^3 \left. \right\}. \quad (7.1)
\end{aligned}$$

We can see the final expression for the energy flux [Eq. (7.1)] is independent of the arbitrary length scale  $z_0$ . Similarly by using Eqs. (6.12) and (5.4) [(5.8)], one can find the complete 3PN expression for energy flux in SH [MH] coordinates. Given the total energy flux as a function of the separation between the two objects at any instant the total energy radiated during the infall can be computed as,

$$\Delta E = - \int_{z_f}^{z_i} \left( \frac{d\mathcal{E}}{dt} \right) \frac{dz}{\dot{z}}, \quad (7.2)$$

where  $z_i$  and  $z_f$  are the initial and final separation between the two objects under head-on infall. Inserting  $s = z/z_i$  and  $\gamma = Gm/c^2 z$  back in Eq. (7.1) and then using it along with  $\dot{z}$  in ADM coordinates in Eq. (7.2) one can compute the total energy radiated during the radial infall of the two objects from a initial separation  $z_i$  to a final separation  $z_f$ . Since Eq. (7.1) involves some integrals which can only be evaluated numerically, we use the NIntegrate option inbuilt in *Mathematica* to compute the total radiated energy during the process of infall. On the other hand for the case of infall from infinity, since we have computed the energy flux as a function of the separation between the two objects

in closed form we shall provide 3PN expression for the total energy radiated during the radial infall from  $z_i = \infty$  to the final separation  $z_f$  in Sec. VII B, however we wish to plot the curves corresponding to the limit  $z_i = \infty$  with those corresponding to the case of infall from a finite distance for comparing the results.

### B. Case II: Infall from infinity

For this case the complete 3PN expression for energy flux in ADM coordinates can be obtained by adding hereditary part (Eq. (6.21)) and instantaneous part (Eq. (5.13)) of energy flux and we have,

$$\begin{aligned} \left(\frac{d\mathcal{E}}{dt}\right)_{\text{ADM}} = & \frac{16}{15} \frac{c^5}{G} \nu^2 \gamma^5 \left\{ 1 + \left[ -\frac{43}{7} + \frac{111}{14} \nu \right] \gamma + \sqrt{2} \left[ -\frac{71}{6} - \frac{5}{\sqrt{3}} \pi - 5 \log \left( \frac{2}{3} \gamma \right) \right] \gamma^{3/2} + \left[ -\frac{4643}{108} - \frac{713}{36} \nu + \frac{112}{3} \nu^2 \right] \gamma^2 \right. \\ & + \sqrt{2} \left[ -\frac{6935}{84} + \frac{2539}{14\sqrt{3}} \pi + \left( -\frac{83953}{840} - \frac{801\sqrt{3}}{28} \pi \right) \nu + \left( \frac{691}{14} - \frac{1507}{28} \nu \right) \log \left( \frac{2}{3} \gamma \right) \right] \gamma^{5/2} \\ & + \left[ \frac{363674263}{582120} + \frac{9433}{42\sqrt{3}} \pi + \frac{145}{6} \pi^2 + \left( -\frac{5467459}{5544} - \frac{1125}{32} \pi^2 \right) \nu + \frac{1865377}{5544} \nu^2 \right. \\ & \left. \left. + \frac{231520}{2079} \nu^3 + \left( \frac{9433}{42} + \frac{65\pi}{\sqrt{3}} \right) \log \left( \frac{2}{3} \gamma \right) + \frac{65}{2} \log^2 \left( \frac{2}{3} \gamma \right) + 80\psi^{(1)} \left( \frac{11}{3} \right) \right] \gamma^3 \right\}. \end{aligned} \quad (7.3)$$

We can now see the final expression for the energy flux [Eq. (7.3)] is independent of the arbitrary length scale  $z_0$ . Similarly by employing Eq. (6.21) with Eq. (5.5) and Eq. (5.9) one finds the complete 3PN expression for energy flux in SH and MH coordinates, respectively.

Given total energy flux as a function of the separation between the two objects at any instant the total energy radiated during the infall can be computed as

$$\Delta E = - \int_{z_f}^{+\infty} \left( \frac{d\mathcal{E}}{dt} \right) \frac{dz}{\dot{z}}. \quad (7.4)$$

Using the expression for energy flux in ADM coordinates given by Eq. (7.3) and  $\dot{z}$  in ADM coordinates given by Eq. (3.21) in the above we get the 3PN expression for total energy radiated due to head-on infall of two compact objects from infinity to a final separation of  $z_f$  as

$$\begin{aligned} \Delta E_{\text{ADM}} = & \frac{16\sqrt{2}}{105} \nu^2 m c^2 \gamma_f^{7/2} \left\{ 1 + \left[ -\frac{17}{6} + \frac{187}{36} \nu \right] \gamma_f + \frac{1}{\sqrt{2}} \left[ -\frac{91}{6} - \frac{7}{\sqrt{3}} \pi - 7 \log \left( \frac{2}{3} \gamma_f \right) \right] \gamma_f^{3/2} \right. \\ & + \left[ -\frac{10508}{297} + \frac{2191}{396} \nu + \frac{18323}{1056} \nu^2 \right] \gamma_f^2 + \frac{1}{\sqrt{2}} \left[ -138 + \frac{197}{\sqrt{3}} \pi + \left( -\frac{5443}{60} - \frac{557}{6\sqrt{3}} \pi \right) \nu \right. \\ & + \left( 43 - \frac{111}{2} \nu \right) \log \left( \frac{2}{3} \gamma_f \right) \left. \right] \gamma_f^{5/2} + \left[ \frac{1183646333}{4684680} + \frac{9013}{78\sqrt{3}} \pi + \frac{1015}{78} \pi^2 \right. \\ & + \left( -\frac{32327629}{61776} - \frac{15883}{832} \pi^2 \right) \nu + \frac{20519431}{82368} \nu^2 + \frac{17017307}{494208} \nu^3 + \left( \frac{9013}{78} + \frac{35}{\sqrt{3}} \pi \right) \log \left( \frac{2}{3} \gamma_f \right) \\ & \left. \left. + \frac{35}{2} \log^2 \left( \frac{2}{3} \gamma_f \right) + \frac{560}{13} \psi^{(1)} \left( \frac{11}{3} \right) \right] \gamma_f^3 \right\}, \end{aligned} \quad (7.5)$$

where  $\gamma_f = G m / c^2 z_f$ .

## VIII. DISCUSSIONS AND CONCLUSION

Having listed the complete 3PN expressions for the GW energy flux [Eq. (7.1) and (7.3)] in ADM coordinates, in this final section we examine its general behavior as a function of the separation between the two objects under the radial infall. Figure 1 shows the variation of the energy flux, in units of  $\nu^2$  scaled by a factor  $c^5/G = 3.63 \times 10^{52}$  joules-sec $^{-1}$ , as a function of the parameter  $\gamma$  in ADM coordinates (recall  $\gamma = G m / c^2 z$  where  $z$  is the instantaneous separation between the two compact objects falling radially towards each other). Each panel in Fig. 1 shows a comparison between the energy flux emitted as a function of the parameter  $\gamma$  for different initial separations including the limiting case of infinite initial separation as well. In each panel curves corresponding to different initial separations (characterized by



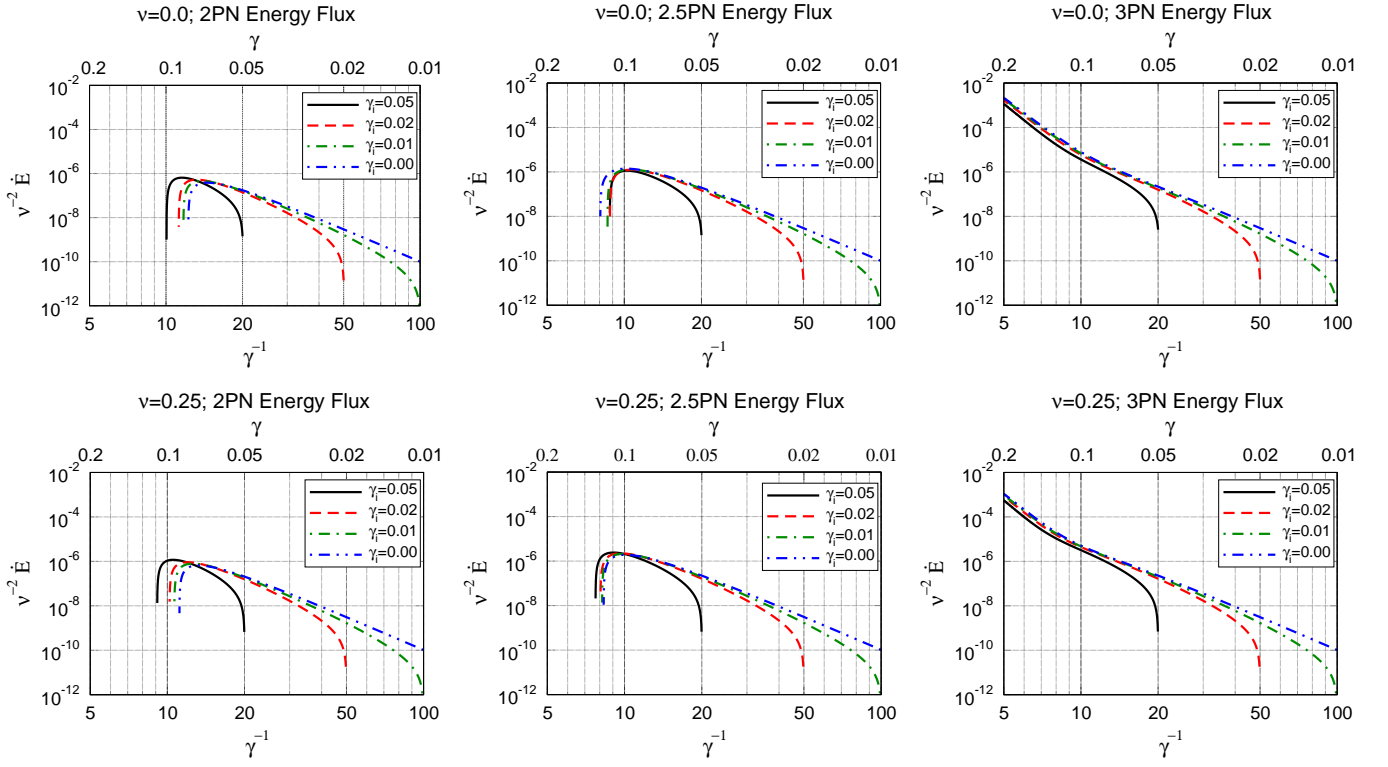


FIG. 1: Energy flux in ADM coordinates, in units of  $\nu^2$  (where  $\nu$  is the symmetric mass ratio of the binary), as a function of the parameter  $\gamma = Gm/c^2 z$  for the head-on situation, for four different initial separations characterized by the parameter  $\gamma_i = Gm/c^2 z_i$ :  $\gamma_i = 0.05$ ,  $\gamma_i = 0.02$ ,  $\gamma_i = 0.01$  and  $\gamma_i = 0.0$  (infinite initial separation limit) which correspond to the situations when the initial separation  $z_i$  between the two objects is  $20 Gm/c^2$ ,  $50 Gm/c^2$ ,  $100 Gm/c^2$  and  $\infty$  respectively. Curves in the top panels correspond to the value of  $\nu = 0$  (test-body limit) while those in the bottom panels correspond to the value of  $\nu = 0.25$  (equal-mass case). Left, middle and right panels in both top and bottom panels correspond to the 2PN, 2.5PN, and 3PN accurate expressions for energy flux. The values given on the y-axis have been scaled by the factor  $c^5/G = 3.63 \times 10^{52} \text{ joules-sec}^{-1}$ . The labels on the x-axis and alternative x-axis corresponds to inverse of the parameter  $\gamma$  (which is the separation between the objects under radial infall at any instant in units of  $Gm/c^2$ ) and the values of the PN parameter  $\gamma$ , respectively.

the parameter  $\gamma_i = Gm/c^2 z_i$ ) have been plotted for  $\gamma_i = 0.05, 0.02, 0.01$ , and  $0.0$  and correspond to the situation when the initial separation  $z_i$  between the two objects is  $20 Gm/c^2$ ,  $50 Gm/c^2$ ,  $100 Gm/c^2$  and  $\infty$  respectively. Curves in the top panels correspond to  $\nu = 0$  (test-body limit) while those in the bottom panels correspond to  $\nu = 0.25$  (equal-mass case). It is obvious from the figure that the curves in each panel approach each other with increasing  $\gamma$  i.e. when the separation between the two objects decreases. This feature can be understood by recalling that since  $s = z/z_i = \gamma_i/\gamma$ , for a fixed  $z_i$ , the finite-separation corrections in powers of  $s$  become progressively less important as the bodies approach each other (small  $z$ ). The finite-separation effects, important when the objects are far apart, are less significant at closer separation and the curves for the energy flux approach each other.

Figure 1 also compares the results that would be obtained using the 2PN, 2.5PN and 3PN accurate expression for the energy flux and thus illustrates the improvements arising from a more accurate expression for the energy flux. It is clear from Fig. 1 that the energy flux emitted at any instant monotonically increases as the separation between the objects under the infall decreases (with increasing  $\gamma$ ) as generally expected. However from Fig. 1 we see that after a certain maximum value of the parameter  $\gamma$  in the 2PN and 2.5PN cases the curves show a turnover and start to decrease. This is an indication of the fact that the PN approximation is no longer valid beyond this value of  $\gamma$ . It should be noted that the value of  $\gamma$  where this happens depends upon the choice of the initial separation between the two objects, the PN accuracy of the expression for the energy flux and the symmetric mass ratio of the binary.

Finally, Fig. 2 shows the total energy radiated [as discussed in the previous section for the finite initial separation case it has to be computed numerically using Eq. (7.2) but for the infinite initial separation case it is given by Eq. (7.4)] during a radial infall from initial separation  $z_i$  (characterized by the parameter  $\gamma_i = Gm/c^2 z_i$ ) to a final separation  $z_f$  (characterized by the parameter  $\gamma_f = Gm/c^2 z_f$ ). Similar to Fig. 1 in Fig. 2 we study the effect of using different PN-accurate expressions for energy flux and also the effect of assuming different initial separations in the problem.

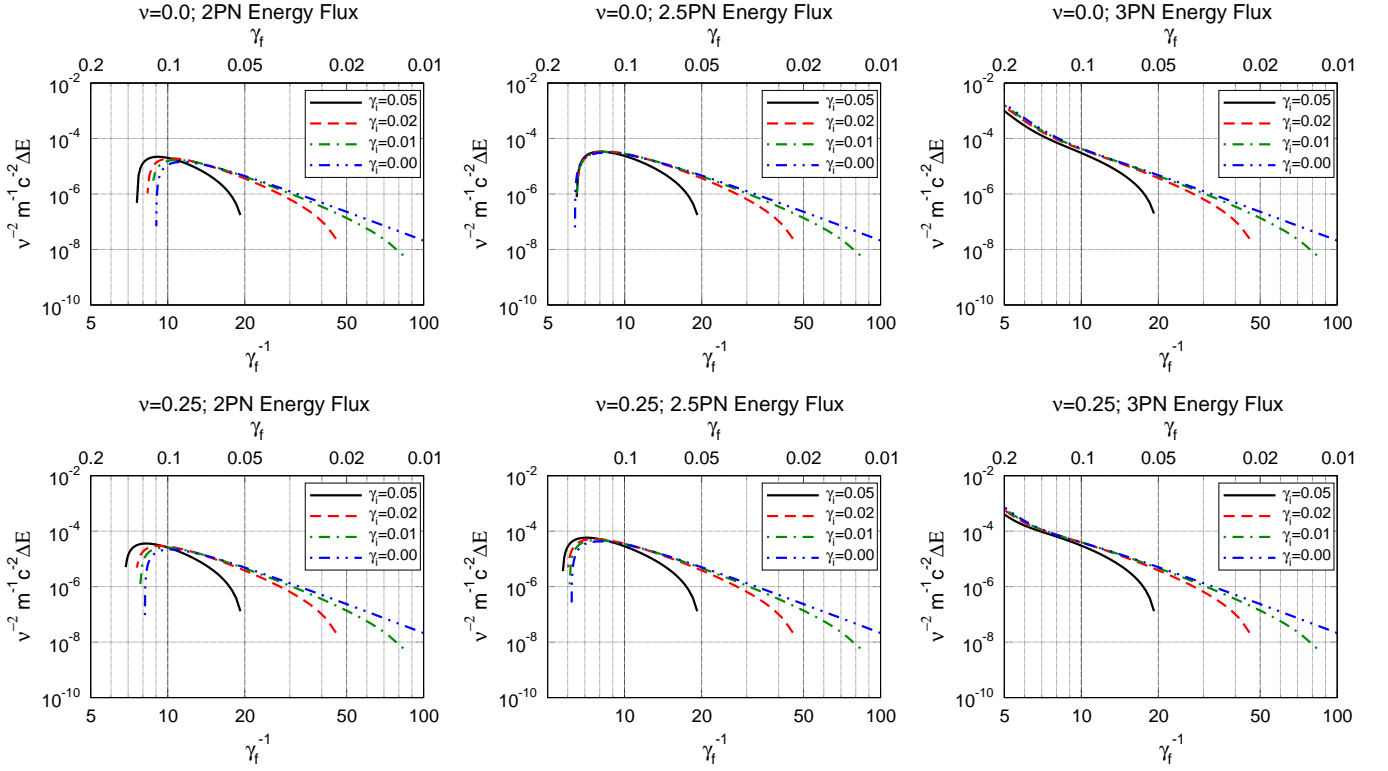


FIG. 2: Similar to Fig. 1 but for total energy radiated in units of  $\nu^2 c^2 m$ , during the head-on infall of two compact objects from a initial separation  $z_i$  in ADM coordinates (related to the parameter  $\gamma_i$ ) to a final separation of  $z_f$  (corresponding parameter  $\gamma_f = G m/c^2 z_f$ ).

It is evident from each panel of the Fig. 2 that as  $\gamma_f$  ( $z_f$ ) increases (decreases) all curves approach each other which implies that most of the contribution comes from the late stages of the infall. It is evident from the plots in Fig. 2 that only beyond a certain minimum separation between the two objects (under the infall) the estimates of energy radiated using PN expressions are valid. The 2PN, 2.5PN and 3PN estimates of the total energy radiated during the radial infall (from infinity) of two equal mass compact objects is of the order of  $2.2 \times 10^{-5}$ ,  $4.3 \times 10^{-5}$  and  $7.4 \times 10^{-5}$  respectively. In the test particle limit The corresponding 2PN, 2.5PN and 3PN accurate results for total energy radiated in the test particle limit are of the order of  $1.4 \times 10^{-5}$ ,  $3.1 \times 10^{-5}$  and  $8.5 \times 10^{-5}$  respectively. Unlike the 2PN and 2.5PN cases where the breakdown of the PN approximation is explicit in the turnover, the 3PN approximation does not show any sharp turnover. As a consequence the value quoted for the maximum energy radiated in the 3PN case is a bit arbitrary and corresponds to the value at the point where the 2.5PN approximation breaks down. From the Fig. 2 one can infer that the energy radiated in the process of head-on infall for the finite separation cases ( $\gamma_i = 0.05, 0.02, 0.01$ ) is of the same order as in infinite initial separation case ( $\gamma_i = 0$ ). It is evident from the above discussion that the 3PN estimates of the peak luminosities and the energy loss in form gravitational radiation during the infall between the initial ( $z_i$ ) and a final point ( $z_f$ ) will not only be more than the estimates of the same using a less accurate expressions (2PN and 2.5PN accurate) but also they are valid till later stages of the infall and thus allows one to compare the results obtained using numerical relativity within the range in which PN approximations are valid.

#### Appendix A: Calculation of $\delta_{(\text{SH} \rightarrow \text{ADM})} E$

General expression for energy  $E$  in CM frame associated with SH coordinate system is given in terms of the natural variables;  $r$ ,  $v$  and  $\dot{r}$  [28]. Noticing this functional dependence and the fact that it is a scalar quantity we expect that under a transformation ( $r' \rightarrow r + \delta r$ ,  $v' \rightarrow v + \delta v$ ,  $\dot{r}' \rightarrow \dot{r} + \delta \dot{r}$ ) this would transform in CM frame as

$$E' = E + \delta E \quad (\text{A1})$$

Or equivalently for transformations between SH and ADM coordinate systems,

$$E_{\text{ADM}} = E_{\text{SH}} + \delta_{(\text{SH} \rightarrow \text{ADM})} E, \quad (\text{A2})$$

where,

$$\delta_{(\text{SH} \rightarrow \text{ADM})} E = \delta r \frac{\partial E}{\partial r} + \delta v \frac{\partial E}{\partial v} + \delta \dot{r} \frac{\partial E}{\partial \dot{r}}. \quad (\text{A3})$$

The shifts in the variables  $r$ ,  $v$  and  $\dot{r}$  connecting ADM and SH coordinates are given by Eq. (6.10) of [25] and the expression for CM energy  $E_{\text{SH}}$  for general orbits is given by Eq. (4.8) of [28]. Having all inputs we now can write the shift  $\delta_{(\text{ADM} \rightarrow \text{SH})} E$  for general orbits which reads as

$$\begin{aligned} \delta_{(\text{SH} \rightarrow \text{ADM})} E = & \frac{G m^2 \nu}{c^4 r} \left[ -\frac{13}{8} \nu v^4 + \frac{5}{4} \nu v^2 \dot{r}^2 + \frac{G m}{r} v^2 \left( \frac{1}{4} + \frac{47}{8} \nu \right) \right. \\ & + \frac{3}{8} \nu \dot{r}^4 + \frac{G m}{r} \dot{r}^2 \left( -\frac{1}{2} - \frac{57}{8} \nu \right) + \frac{G^2 m^2}{r^2} \left( \frac{1}{4} + 3 \nu \right) \Big] \\ & + \frac{G m^2 \nu}{c^6 r} \left[ v^6 \left( -\frac{65}{16} \nu + \frac{179}{16} \nu^2 \right) + \dot{r}^2 v^4 \left( \frac{61}{16} \nu - \frac{165}{16} \nu^2 \right) \right. \\ & + \frac{G m}{r} v^4 \left( \frac{3}{8} + \frac{7}{8} \nu - \frac{481}{16} \nu^2 \right) + \dot{r}^4 v^2 \left( \frac{9}{16} \nu - \frac{39}{16} \nu^2 \right) + \frac{G m}{r} \dot{r}^2 v^2 \left( -\frac{3}{4} - \frac{131}{16} \nu + \frac{641}{16} \nu^2 \right) \\ & + \frac{G^2 m^2}{r^2} v^2 \left( \frac{3}{8} + \left[ \frac{26167}{1680} - \frac{21 \pi^2}{32} + \frac{22}{3} \log \left( \frac{r}{r'_0} \right) \right] \nu + \frac{37}{8} \nu^2 \right) + \dot{r}^6 \left( -\frac{5}{16} \nu + \frac{25}{16} \nu^2 \right) \\ & + \frac{G m}{r} \dot{r}^4 \left( \frac{671}{48} \nu - 6 \nu^2 \right) + \frac{G^2 m^2}{r^2} \dot{r}^2 \left( -\frac{3}{2} + \left[ \frac{6479}{1680} + \frac{63 \pi^2}{32} - 22 \log \left( \frac{r}{r'_0} \right) \right] \nu - \frac{23}{2} \nu^2 \right) \\ & \left. + \frac{G^3 m^3}{r^3} \left( -\frac{1}{4} + \left[ -\frac{3613}{280} - \frac{21 \pi^2}{32} + \frac{22}{3} \log \left( \frac{r}{r'_0} \right) \right] \nu \right) \right] \quad (\text{A4}) \end{aligned}$$

It is easy to see that when restrictions given by Eq. (3.1) are imposed, the above expression reduces to the form given by Eq. (3.17).

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